Binomial matrices for polynomials calculating sums of powers with bases in arithmetic progression

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Traslation [3]

1 A new solution

The classical problem of the sum of powers of successive integers is generalized to sums of powers having for base any arithmetic progressions and solved by a simple formula, found and proved by the author, through the product of two matrices closely related binomial coefficients. The result is then connected with the traditional formula of Faulhaber suitably generalized by means of Bernoulli polynomials.

1.1 Warnings

To simplify the notation we consider $0^0 = 1$
We will only consider lower triangular matrices and therefore of we will often omit to repeat that the elements above the upper diagonal they are null. We will use the integers r and c for rows and columns often implying the variability from 1 to m, order of the matrix.

2 Definition

2.1 Vectors

Definition 1 ***Vandermonde vectors***
Let $\vec{V}(j)$ be the vector with m composed defined as follows:

$$[\vec{V}(j)]_r = j^{r-1} \text{ with } r = 1, 2, 3, ...m \quad r, m \in \mathbb{N}^+ \quad j \in \mathbb{C}$$

It is applied in: D:2; P:1,2,3,4

Definition 2 ***Sums of Vandermonde vectors in arithmetic progression***
Vector with m : $n \in \mathbb{N}^+ \quad h, d \in \mathbb{C}$

$$\vec{S}_{h,d}(n) = \sum_{k=0}^{n-1} \vec{V}(h + dk) \qquad [\vec{S}_{h,d}(n)]_r = \sum_{k=0}^{n-1} (h + dk)^{r-1} \text{ with } r\text{=}1...m$$

It applies: D:1 It is applied in: P:2,4

2.2 Matrices related to the Pascal's triangle

Definition 3 ***Matrix A*** *Triangle Matrix without last row element*

$$[A]_{r,c} = \binom{r}{c-1} \quad \textit{if } c \le r, \quad \textit{otherwise } 0$$

It is applied in: P:1,2,4,5

Definition 4 ***T(h,d)*** *Non-abelian group of binomial matrices of order m*

$$[T(h,d)]_{r,c} = \binom{r-1}{c-1} h^{r-c} d^{c-1} \quad \textit{if } c \le r, \quad \textit{otherwise } 0$$

It is applied in P:3,4,5

Definition 5 ***G(h,d)*** *Matrices with Bernoulli polynomials of order m*

$$[G(h,d)]_{r,c} = \frac{d^{r-1}}{r} \binom{r}{c} B_{r-c}\left(\frac{h}{d}\right) \quad \textit{if } c \le r, \ \textit{otherwise } 0$$

It is applied in P:3,4,5

3 Proposition

Proposition 1 ***Identità G0***

$$A\vec{V}(k) = (1+k)\vec{V}(1+k) - k\vec{V}(k)$$

It applies: D:1,3 It is applied in: P:2
Taking into account that due to the triangularity of the matrix the result is $[A]_{r,j} = 0$ se $j > r$, multiplying row by column, we have:

$$\sum_{j=1}^{m} [A]_{r,j} [\vec{V}(k)]_j = \sum_{j=1}^{r} \binom{r}{j-1} k^{j-1} = -k^r + \sum_{j=1}^{r+1} \binom{r}{j-1} k^{j-1} = (k+1)^r - k^r =$$

$$= [(k+1)\vec{V}(k+1)]_r - [k\vec{V}(k)]_r \qquad \text{q.e.d.}$$

3.1 Zero G theorem

Proposition 2 ***Theorem G0*** *Solves the problem of the sum of powers of successive integers starting from* 0*.*

$$\vec{S}_{0,1}(n) = A^{-1} n \vec{V}(n)$$

It applies: D:1,2,3; P:1 It is applied in: P:4

Adding member to member, starting from 0, the first n particular cases of P:1 we obtain: $\sum_{k=0}^{n-1} A\vec{V}(k) = \sum_{k=0}^{n-1}\bigg((1+k)\vec{V}(1+k) - k\vec{V}(k)\bigg)$ By developing the sum on the second member almost all the terms are simplified (telescopic effect). Then collecting the matrix of the first member as a common factor we obtain: $A\sum_{k=0}^{n-1}\vec{V}(k) = n\vec{V}(n) - 0\vec{V}(0)$ from which omitting the vector subtracting which is zero we obtain: $A\sum_{k=0}^{n-1}\vec{V}(k) = n\vec{V}(n)$ and finally, multiplying the members of the equation on the left by A^{-1} (det A$= m! \neq 0$) we gets the thesis. q.e.d.

Proposition 3 ***Matrices for linear transformation***

$$T(h,p)\vec{V}(x) = \vec{V}(h+px)$$

It applies: D:1,4 It is applied in: P:4

Performing the product rows by columns $\sum_{k=1}^{m}[T(h,p)]_{r,k}[\vec{V}(x)]_k$:
replacing according to the definitions given and remembering that, for triangularity of the matrix, $[T(h,p)]_{r,k} = 0$ per $k > r$ we obtain $\sum_{k=1}^{r}\binom{r-1}{k-1}h^{r-k}p^{k-1}x^{k-1} = \sum_{k=1}^{r}\binom{r-1}{k-1}h^{r-k}(px)^{k-1} = (h+px)^{r-1} = [\vec{V}(h+px)]_r$ q.e.d.

3.2 G theorem

Proposition 4 ***G theorem***

$$\vec{S}_{h,d}(n) = T(h,d)A^{-1}\ n\vec{V}(n)$$

It applies: D:1,2,3,4; P:2,3 It is applied in:

The statement of this proposition is obtained from $\vec{S}_{0,1}(n) = A^{-1}n\vec{V}(n)$ (P:2) multiplying the two left sides by the matrix $T(h,d)$. Indeed:

$$T(h,d)\vec{S}_{0,1}(n) = T(h,d)\sum_{k=0}^{n-1}\vec{V}(k) = \sum_{k=0}^{n-1}T(h,d)\vec{V}(k) = \sum_{k=0}^{n-1}\vec{V}(h+dk) = \vec{S}_{h,d}(n)$$

q.e.d.

3.3 Connection with Faulhaber's formula

Proposition 5 *Equivalence for lower triangular matrices of order m:*

$$G(h,d) = T(h,d)A^{-1}$$

It applies: D:3,4,5;

Multiplying the two sides of the equality by A on the right we obtain the equivalent equation $G(h,d)A = T(h,d)$ which we will prove. Given the components of G (D:5) and of A (D:3), taking into account the null terms due to the lower triangular matrices, the product r-th row by c-th column ($r = 1...m;\ c = 1...m$) is $\sum_{k=1}^{m}[G(h,d)]_{r,k}[A]_{k,c} = \sum_{k=c}^{r}\frac{d^{r-1}}{r}\binom{r}{k}B_{r-k}(\frac{h}{d})\binom{k}{c-1}$ if $r \geq c$ otherwise 0. Developing we have:

$$= \frac{d^{r-1}}{r}\sum_{k=c}^{r}\frac{r!}{k!(r-k)!}\frac{k!}{(c-1)!(k-c+1)!}B_{r-k}\left(\frac{h}{d}\right) =$$

$$= \frac{d^{r-1}}{r} \sum_{k=c}^{r} \frac{r!}{(c-1)!} \frac{1}{(r-k)!(k-c+1)!} B_{r-k}(\frac{h}{d}\Big) =$$

$$= \frac{d^{r-1}}{r} \sum_{k=c}^{r} \frac{r!}{(c-1)!(r-c+1)!} \frac{(r-c+1)!}{(r-k)!(k-c+1)!} B_{r-k}\left(\frac{h}{d}\right) =$$

$$= \frac{d^{r-1}}{r} \sum_{k=c}^{r} \binom{r}{c-1} \binom{r-c+1}{r-k} B_{r-k}\left(\frac{h}{d}\right) =$$

$$= \frac{d^{r-1}}{r} \binom{r}{c-1} \sum_{k=c}^{r} \binom{r-c+1}{r-k} B_{r-k}\left(\frac{h}{d}\right) =$$

$$= \frac{d^{r-1}}{r} \binom{r}{c-1} \sum_{q=0}^{r-c} \binom{r-c+1}{q} B_q\left(\frac{h}{d}\right) = \frac{d^{r-1}}{r} \binom{r}{c-1} (r-c+1) \left(\frac{h}{d}\right)^{r-c}$$

$$= \frac{d^{r-1}}{r} \frac{r!(r-c+1)}{(c-1)!(r-c+1)!} \frac{h^{r-c}}{d^{r-c}} = \binom{r-1}{c-1} h^{r-c} d^{c-1} = [T(h,d)]_{r,c}$$

In fact, in the factor resulting from the summation where we have set $q = r - k$ taking into account the inversion property of Bernoulli polynomials so:

$$(n+1)x^n = \sum_{k=0}^{n} \binom{n+1}{k} B_k(x)$$

with $n = r - c$ and $x = \frac{h}{d}$ q.e.d.

Example 1 $G(h,d) = T(h,d)A^{-1}$ ***caso*** $m = 5$.

$$= T(h,d)A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ h & d & 0 & 0 & 0 \\ h^2 & 2hd & d^2 & 0 & 0 \\ h^3 & 3h^2d & 3hd^2 & d^3 & 0 \\ h^4 & 4h^3d & 6h^2d^2 & 4hd^3 & d^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 \\ 1 & 5 & 10 & 10 & 5 \end{bmatrix}^{-1} =$$

$$= G(h,d) = \begin{bmatrix} 1\frac{1}{1}B_0(\frac{h}{d}) & 0 & 0 & 0 & 0 \\ 2\frac{d}{2}B_1(\frac{h}{d}) & 1\frac{d}{2}B_0(\frac{h}{d}) & 0 & 0 & 0 \\ 3\frac{d^2}{3}B_2(\frac{h}{d}) & 3\frac{d^2}{3}B_1(\frac{h}{d}) & 1\frac{d^2}{3}B_0(\frac{h}{d}) & 0 & 0 \\ 4\frac{d^3}{4}B_3(\frac{h}{d}) & 6\frac{d^3}{4}B_2(\frac{h}{d}) & 4\frac{d^3}{4}B_1(\frac{h}{d}) & 1\frac{d^3}{4}B_0(\frac{h}{d}) & 0 \\ 5\frac{d^4}{5}B_4(\frac{h}{d}) & 10\frac{d^4}{5}B_3(\frac{h}{d}) & 10\frac{d^4}{5}B_2(\frac{h}{d}) & 5\frac{d^4}{5}B_1(\frac{h}{d}) & 1\frac{d^4}{5}B_0(\frac{h}{d}) \end{bmatrix}$$

It applies: D:3,4,5; P:5

Example 2 ***G theorem: Sum of powers of successive odd numbers****.*
$h = 1$, $d = 2$, order $m = 10$. To apply the G theorem one must consider:

$$T(1,2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 12 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 8 & 24 & 32 & 16 & 0 & 0 & 0 & 0 & 0 \\ 1 & 10 & 40 & 80 & 80 & 32 & 0 & 0 & 0 & 0 \\ 1 & 12 & 60 & 160 & 240 & 192 & 64 & 0 & 0 & 0 \\ 1 & 14 & 84 & 280 & 560 & 672 & 448 & 128 & 0 & 0 \\ 1 & 16 & 112 & 448 & 1120 & 1792 & 1792 & 1024 & 256 & 0 \\ 1 & 18 & 144 & 672 & 2016 & 4032 & 5376 & 4608 & 2304 & 512 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 \end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & -\frac{7}{24} & 0 & \frac{1}{12} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{2}{9} & 0 & -\frac{7}{15} & 0 & \frac{2}{3} & -\frac{1}{2} & \frac{1}{9} & 0 \\ 0 & -\frac{3}{20} & 0 & \frac{1}{2} & 0 & -\frac{7}{10} & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{10} \end{bmatrix}$$

Once the matrix $G_{1,2} = T(1,2)A^{-1}$ we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{7}{15} & 0 & -\frac{8}{3} & 0 & \frac{16}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{3} & 0 & -\frac{20}{3} & 0 & \frac{16}{3} & 0 & 0 & 0 & 0 \\ -\frac{31}{21} & 0 & \frac{28}{3} & 0 & -16 & 0 & \frac{64}{7} & 0 & 0 & 0 \\ 0 & -\frac{31}{3} & 0 & \frac{98}{3} & 0 & -\frac{112}{3} & 0 & 16 & 0 & 0 \\ \frac{127}{15} & 0 & -\frac{496}{9} & 0 & \frac{1568}{15} & 0 & -\frac{256}{3} & 0 & \frac{256}{9} & 0 \\ 0 & \frac{381}{5} & 0 & -248 & 0 & \frac{1568}{5} & 0 & -192 & 0 & \frac{256}{5} \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ n^9 \\ n^{10} \end{bmatrix} =$$

performing the product rows by column:

$$= \begin{bmatrix} \sum_{k=0}^{n-1}(1+2k)^0 \\ \sum_{k=0}^{n-1}(1+2k)^1 \\ \sum_{k=0}^{n-1}(1+2k)^2 \\ \sum_{k=0}^{n-1}(1+2k)^3 \\ \sum_{k=0}^{n-1}(1+2k)^4 \\ \sum_{k=0}^{n-1}(1+2k)^5 \\ \sum_{k=0}^{n-1}(1+2k)^6 \\ \sum_{k=0}^{n-1}(1+2k)^7 \\ \sum_{k=0}^{n-1}(1+2k)^8 \\ \sum_{k=0}^{n-1}(1+2k)^9 \end{bmatrix} = \begin{bmatrix} n \\ n^2 \\ -\frac{1}{3}n + \frac{4}{3}n^3 \\ -n^2 + 2n^4 \\ \frac{7}{15}n - \frac{8}{3}n^3 + \frac{16}{5}n^5 \\ \frac{7}{3}n^2 - \frac{20}{3}n^4 + \frac{16}{3}n^6 \\ -\frac{31}{21}n + \frac{28}{3}n^3 - 16n^5 + \frac{64}{7}n^7 \\ -\frac{31}{3}n^2 + \frac{98}{3}n^4 - \frac{112}{3}n^6 + 16n^8 \\ \frac{127}{15}n - \frac{496}{9}n^3 + \frac{1568}{15}n^5 - \frac{256}{3}n^7 + \frac{256}{9}n^9 \\ \frac{381}{5}n^2 - 248n^4 + \frac{1568}{5}n^6 - 192n^8 + \frac{256}{5}n^{10} \end{bmatrix}$$

Applica: D:3,4; P:4

Example 3 ***G theorem: sums of powers of the arithmetic progression 3k+1.***

We now use the G theorem to calculate sums of powers of divided integers for three they give remainder 1. It turns out $h = 1,\ d = 3$. *We choosem* $= 4$ *limiting ourselves to first 4 polynomials. To apply the G theorem one must consider:*

$$T(1,3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 6 & 9 & 0 \\ 1 & 9 & 27 & 27 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 4 & 6 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Having obtained the matrix $G_{1,3} = T(1,3)A^{-1}$ *we have:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{3}{2} & 3 & 0 \\ 1 & -\frac{9}{4} & -\frac{9}{2} & \frac{27}{4} \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \end{bmatrix} =$$

performing the product rows by column:

$$= \begin{bmatrix} \sum_{k=0}^{n-1}(1+3k)^0 \\ \sum_{k=0}^{n-1}(1+3k)^1 \\ \sum_{k=0}^{n-1}(1+3k)^2 \\ \sum_{k=0}^{n-1}(1+3k)^3 \end{bmatrix} = \begin{bmatrix} n \\ -\frac{1}{2}n + \frac{3}{2}n^2 \\ -\frac{1}{2}n - \frac{3}{2}n^2 + 3n^3 \\ n - \frac{9}{4}n^2 - \frac{9}{2}n^3 + \frac{27}{4}n^4 \end{bmatrix}$$

It applies: D:3,4; P:4

References

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