On polynomials for the calculation of sums of powers of successive integers and Bernoulli numbers deduced from the Pascal's triangle

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Abstract: In the "twin" theorems 1A and 1B, two matrices derived from the Pascal's triangle, the first of which has alternate signs, are taken into account, and it is shown that their inverse gives, with a minimum variation in the second case, the coefficients of famous polynomials for calculating the sums of powers of successive integers (fig.3). These coefficients are determined here without the help of Bernoulli's numbers. In the theorems 2A and 2B, these famous numbers are obtained starting from other matrices closely related to the previous ones.

1.Theorem 1A. It will be shown that the polynomial coefficients for calculating the sum of powers of n successive integers can be obtained by inverting a particular alternate-sign matrix in relationship to Pascal's triangle.

Suppose A_m is a square matrix of order m defined as:

$$a_{j,k} = \begin{cases} 0 & \text{if } k > j \\ \binom{j}{k-1} * (-1)^{j+k} & \text{otherwise} \end{cases}$$

This is a triangular matrix where the triangle of Tartaglia appears, with alternate signs, private of the last element of each line. Its main diagonal is therefore formed by the succession of the positive integers whose product (m!) gives the determinant.

Therefore:

$$A_m * \begin{pmatrix} \sum_{i=1}^n i^0 \\ \sum_{i=1}^n i^1 \\ \sum_{i=1}^n i^2 \\ \dots \\ \sum_{i=1}^n i^{m-1} \end{pmatrix} = A_m * \sum_{i=1}^{n} \begin{pmatrix} i^0 \\ i^1 \\ i^2 \\ \dots \\ i^{m-1} \end{pmatrix} = \sum_{i=1}^{n} A_m * \begin{pmatrix} i^0 \\ i^1 \\ i^2 \\ \dots \\ i^{m-1} \end{pmatrix} =$$

$$= \sum_{i=1}^{n} \begin{pmatrix} i^1 - (i-1)^1 \\ i^2 - (i-1)^2 \\ i^3 - (i-1)^3 \\ \dots \\ i^m - (i-1)^m \end{pmatrix} = \begin{pmatrix} n^1 \\ n^2 \\ n^3 \\ \dots \\ n^m \end{pmatrix}$$

In the first two steps we applied the distributive property, in the third step multipling the A_m matrix by the vectors of powers, we get:

$$\sum_{k=1}^{m} a_{j,k} * i^{k-1} = \sum_{k=1}^{j} \binom{j}{k-1} * (-1)^{j+k} * i^{k-1} =$$

$$= \sum_{k=1}^{j} \ (-1)^{2k-1} \binom{j}{k-1} * (-1)^{j-k+1} * i^{k-1} =$$

$$= -\sum_{k=1}^{j} \binom{j}{k-1} * (-1)^{j-k+1} * i^{k-1} =$$

$$= -(-i^j + \sum_{k=1}^{j+1} \binom{j}{k-1} * (-1)^{j-k+1} * i^{k-1}) =$$

$$= -(-i^j + (i-1)^j) = i^j - (i-1)^j$$

Therefore we find:

$$A_m * \begin{pmatrix} \sum_{i=1}^{n} i^0 \\ \sum_{i=1}^{n} i^1 \\ \sum_{i=1}^{n} i^2 \\ \dots \\ \sum_{i=1}^{n} i^{m-1} \end{pmatrix} = \begin{pmatrix} n \\ n^2 \\ n^3 \\ \dots \\ n^m \end{pmatrix}$$

Multiplyng the two left hand members by the inverse matrix of $A_m$we get:

$$\boxed{\begin{pmatrix} \sum_{i=1}^{n} i^0 \\ \sum_{i=1}^{n} i^1 \\ \sum_{i=1}^{n} i^2 \\ \dots \\ \sum_{i=1}^{n} i^{m-1} \end{pmatrix} = A_m^{-1} \begin{pmatrix} n \\ n^2 \\ n^3 \\ \dots \\ n^m \end{pmatrix}}$$

Thus, the theorem provides the famous polynomials for computing the sum of powers of successive integers already studied by Faulhaber in the seventeenth century

Example given m=11

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 & 0 & 0 & 0 & 0 \\ -1 & 8 & -28 & 56 & -70 & 56 & -28 & 8 & 0 & 0 & 0 \\ 1 & -9 & 36 & -84 & 126 & -126 & 84 & -36 & 9 & 0 & 0 \\ -1 & 10 & -45 & 120 & -210 & 252 & -210 & 120 & -45 & 10 & 0 \\ 1 & -11 & 55 & -165 & 330 & -462 & 462 & -330 & 165 & -55 & 11 \end{pmatrix}$$

And applying the theorem:

$$\begin{pmatrix} \sum_{i=1}^{n} i^0 \\ \sum_{i=1}^{n} i^1 \\ \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} i^3 \\ \sum_{i=1}^{n} i^4 \\ \sum_{i=1}^{n} i^5 \\ \sum_{i=1}^{n} i^6 \\ \sum_{i=1}^{n} i^7 \\ \sum_{i=1}^{n} i^8 \\ \sum_{i=1}^{n} i^9 \\ \sum_{i=1}^{n} i^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & -\frac{7}{24} & 0 & \frac{7}{12} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{2}{9} & 0 & -\frac{7}{15} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{9} & 0 & 0 \\ 0 & -\frac{3}{20} & 0 & \frac{1}{2} & 0 & -\frac{7}{10} & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{10} & 0 \\ \frac{5}{66} & 0 & -\frac{1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & \frac{1}{2} & \frac{1}{11} \end{pmatrix} * \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ n^9 \\ n^{10} \\ n^{11} \end{pmatrix}$$

To note in the first column of the square matrix the appearance of Bernoulli's original numbers.

Or likewise:

$$\sum_{i=1}^{n} i^0 = n$$

$$\sum_{i=1}^{n} i^1 = \frac{1}{2}n + \frac{1}{2}n^2$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$$

$$\sum_{i=1}^{n} i^4 = -\frac{1}{30}n + \frac{1}{3}n^3 + \frac{1}{2}n^4 + \frac{1}{5}n^5$$

$$\sum_{i=1}^{n} i^5 = -\frac{1}{12}n^2 + \frac{5}{12}n^4 + \frac{1}{2}n^5 + \frac{1}{6}n^6$$

$$\sum_{i=1}^{n} i^6 = \frac{1}{42}n - \frac{1}{6}n^3 + \frac{1}{2}n^5 + \frac{1}{2}n^6 + \frac{1}{7}n^7$$

$$\sum_{i=1}^{n} i^7 = \frac{1}{12}n^2 - \frac{7}{24}n^4 + \frac{7}{12}n^6 + \frac{1}{2}n^7 + \frac{1}{8}n^8$$

$$\sum_{i=1}^{n} i^8 = -\frac{1}{30}n + \frac{2}{9}n^3 - \frac{7}{15}n^4 + \frac{2}{3}n^6 + \frac{1}{2}n^8 + \frac{1}{9}n^9$$

$$\sum_{i=1}^{n} i^9 = -\frac{3}{20}n^2 + \frac{1}{2}n^4 - \frac{7}{10}n^6 + \frac{3}{4}n^8 + \frac{1}{2}n^9 + \frac{1}{10}n^{10}$$

$$\sum_{i=1}^{n} i^{10} = \frac{5}{66}n - \frac{1}{2}n^3 + n^5 - n^7 + \frac{5}{6}n^9 + \frac{1}{2}n^{10} + \frac{1}{11}n^{11}$$

These are the polynomials studied by Faulhaber and Bernoulli

... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$\int n = \frac{1}{2}nn + \frac{1}{2}n$
$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$
$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$
$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$
$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$
$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$
$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$
$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$
$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}nn$
$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c \cdot c-1 \cdot c-2}{2 \cdot 3 \cdot 4}Bn^{c-3}$$
$$+\frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}Cn^{c-5}$$
$$+\frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4 \cdot c-5 \cdot c-6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}Dn^{c-7} \cdots \text{\& ita deinceps,}$$

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

fig.3: Ars Conjectandi di Jakob Bernoulli (1654-1705) Published in 1713. (the monomial of degree 2 of polinomial of degree 10 is wrong)

2.Theorem 2A. It is proved as a corollary of the previous theorem 1A, a formula for obtaining Bernoulli numbers from the Pascal triangle

We have seen that the matrix A_n so defined

$$a_{j,k} = \begin{cases} 0 & \text{if } k > j \\ \binom{j}{k-1} * (-1)^{j+k} & \text{otherwise} \end{cases}$$

has an inverse matrix that has the first-degree coefficients of Faulhaber polynomials in the first column. These correspond to "second Bernoulli numbers" or even "Bernoulli's original numbers" where $B_1 = +\frac{1}{2}$ stands. The most commonly considered "first Bernoulli numbers" whose sequence differs only for $B_1 = -\frac{1}{2}$.

Recalling the Laplace algorithm for the calculation of the inverse matrices we have;

$$c_{n,1} = B_{n-1} = (-1)^{n+1} \frac{|A_{1,n}|}{|A_n|} = (-1)^{n+1} \frac{|A_{1,n}|}{n!}$$

where:

$c_{n,1}$ indicates the corresponding element of $a_{n,1}$ in the reverse matrix

$| A_n | = n!$ Is the determinant of the triangular matrix.

$A_{1,n}$ is the algebraic complement (obtained by deleting first row and last column) relative to the corresponding $a_{1,n}$ element of $a_{n,1}$ in the transposition for the inverse calculation.

It is therefore possible to define Bernoulli numbers as follows:

$$B_n = (-1)^n \frac{|A_n|}{(n+1)!}$$

Where however | A_n | is the determinant of a n order matrix defined as follows:

$$a_{i,k} = \begin{cases} 0 & \text{if } k > 1+i \\ (-1)^{i+k+1}\binom{i+1}{k-1} & \text{otherwise} \end{cases}$$

This formula gives second Bernoulli numbers. If the factor $(-1)^n$, is omitted, since the only Bernoulli number from zero with odd index is B_1, we get the first Bernoulli numbers:

$$B_n = \frac{|A_n|}{(n+1)!}$$

Examples:

$$B_1 = \frac{|-1|}{2!} = -\frac{1}{2}$$

$$B_2 = \frac{\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix}}{3!} = \frac{1}{6}$$

$$B_3 = \frac{\begin{vmatrix} -1 & 2 & 0 \\ 1 & -3 & 3 \\ -1 & 4 & -6 \end{vmatrix}}{4!} = \frac{0}{24} = 0 \qquad B_4 = \frac{\begin{vmatrix} -1 & 2 & 0 & 0 \\ 1 & -3 & 3 & 0 \\ -1 & 4 & -6 & 4 \\ 1 & -5 & 10 & -10 \end{vmatrix}}{5!} = \frac{-4}{120} = -\frac{1}{30}$$

$$B_5 = \frac{\begin{vmatrix} -1 & 2 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 \\ 1 & -5 & 10 & -10 & 5 \\ -1 & 6 & -15 & 20 & -15 \end{vmatrix}}{6!} = \frac{0}{720} = 0$$

$$B_6 = \frac{\begin{vmatrix} -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 \\ 1 & -7 & 21 & -35 & 35 & -21 \end{vmatrix}}{7!} = \frac{120}{5040} = \frac{1}{42}$$

3.Theorem 1B. It will be shown that the polynomial coefficients for calculating the sum of powers of n-1 successive integers can be obtained by inverting a particular alternate-sign matrix in relationship to Pascal's triangle.

Suppose A_m is a square matrix of order m defined as:

$$a_{j,k} = \begin{cases} 0 & \text{if } k > j \\ \binom{j}{k-1} & \text{otherwise} \end{cases}$$

This is a triangular matrix where the triangle of Tartaglia appears private of the last element of each line. Its main diagonal is therefore formed by the succession of the positive integers whose product (m!) gives the determinant.

Therefore:

$$A_m * \begin{pmatrix} 1+\sum_{i=1}^{n-1} i^0 \\ \sum_{i=1}^{n-1} i^1 \\ \sum_{i=1}^{n-1} i^2 \\ \dots \\ \sum_{i=1}^{n-1} i^{m-1} \end{pmatrix} = A_m * \begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} + A_m * \sum_{i=1}^{n} \begin{pmatrix} i^0 \\ i^1 \\ i^2 \\ \dots \\ i^{m-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} + \sum_{i=1}^{n} A_m * \begin{pmatrix} i^0 \\ i^1 \\ i^2 \\ \dots \\ i^{m-1} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} + \sum_{i=1}^{n} \begin{pmatrix} (i+1)^1 - i^1 \\ (i+1)^2 - i^2 \\ (i+1)^3 - i^3 \\ \dots \\ (i+1)^m - i^m \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} + \begin{pmatrix} n^1 - 1 \\ n^2 - 1 \\ n^3 - 1 \\ \dots \\ n^m - 1 \end{pmatrix} = \begin{pmatrix} n \\ n^2 \\ n^3 \\ \dots \\ n^m \end{pmatrix}$$

In the first two steps we applied the distributive property, in the third step multiplyng the A_m matrix by the vectors of powers, we get:

$$\sum_{k=1}^{m} a_{j,k} * i^{k-1} = \sum_{k=1}^{j} \binom{j}{k-1} * i^{k-1} =$$

$$= -i^j + \sum_{k=1}^{j+1} \binom{j}{k-1} * i^{k-1} =$$

$$= -i^j + (i+1)^j = (i+1)^j - i^j$$

Therefore we find:

$$A_m * \begin{pmatrix} 1 + \sum_{i=1}^{n-1} i^0 \\ \sum_{i=1}^{n-1} i^1 \\ \sum_{i=1}^{n-1} i^2 \\ \dots \\ \sum_{i=1}^{n-1} i^{m-1} \end{pmatrix} = \begin{pmatrix} n \\ n^2 \\ n^3 \\ \dots \\ n^m \end{pmatrix}$$

Moltiplicando i due membri a sinistra per l'inversa di A_m otteniamo:
Multiplying the two left hand members of equation by the inverse matrix of $A_m$we get:

$$\boxed{\begin{pmatrix} 1 + \sum_{i=1}^{n-1} i^0 \\ \sum_{i=1}^{n-1} i^1 \\ \sum_{i=1}^{n-1} i^2 \\ \dots \\ \sum_{i=1}^{n-1} i^{m-1} \end{pmatrix} = A_m^{-1} \begin{pmatrix} n \\ n^2 \\ n^3 \\ \dots \\ n^m \end{pmatrix}}$$

Thus, the theorem provides the variant note of the polynomials for calculating the sum of n-1 addendums

Example given m=11

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 0 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 0 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 0 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 0 \\ 1 & 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 & 11 \end{pmatrix}$$

applying the theorem

$$\begin{pmatrix} 1+\sum_{i=1}^{n-1} i^0 \\ \sum_{i=1}^{n-1} i^1 \\ \sum_{i=1}^{n-1} i^2 \\ \sum_{i=1}^{n-1} i^3 \\ \sum_{i=1}^{n-1} i^4 \\ \sum_{i=1}^{n-1} i^5 \\ \sum_{i=1}^{n-1} i^6 \\ \sum_{i=1}^{n-1} i^7 \\ \sum_{i=1}^{n-1} i^8 \\ \sum_{i=1}^{n-1} i^9 \\ \sum_{i=1}^{n-1} i^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & -\frac{7}{24} & 0 & \frac{7}{12} & -\frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{2}{9} & 0 & -\frac{7}{15} & 0 & \frac{2}{3} & -\frac{1}{2} & \frac{1}{9} & 0 & 0 \\ 0 & -\frac{3}{20} & 0 & \frac{1}{2} & 0 & -\frac{7}{10} & 0 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{10} & 0 \\ \frac{5}{66} & 0 & -\frac{1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & -\frac{1}{2} & \frac{1}{11} \end{pmatrix} * \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ n^9 \\ n^{10} \\ n^{11} \end{pmatrix}$$

To note in the first column of the square matrix the appearance of first Bernoulli numbers.

Or likewise:

$$\sum_{i=1}^{n-1} i^0 = n - 1$$

$$\sum_{i=1}^{n-1} i^1 = -\frac{1}{2}n + \frac{1}{2}n^2$$

$$\sum_{i=1}^{n-1} i^2 = \frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3$$

$$\sum_{i=1}^{n-1} i^3 = \frac{1}{4}n^2 - \frac{1}{2}n^3 + \frac{1}{4}n^4$$

$$\sum_{i=1}^{n-1} i^4 = -\frac{1}{30}n + \frac{1}{3}n^3 - \frac{1}{2}n^4 + \frac{1}{5}n^5$$

$$\sum_{i=1}^{n-1} i^5 = -\frac{1}{12}n^2 + \frac{5}{12}n^4 - \frac{1}{2}n^5 + \frac{1}{6}n^6$$

$$\sum_{i=1}^{n-1} i^6 = \frac{1}{42}n - \frac{1}{6}n^3 + \frac{1}{2}n^5 - \frac{1}{2}n^6 + \frac{1}{7}n^7$$

$$\sum_{i=1}^{n-1} i^7 = \frac{1}{12}n^2 - \frac{7}{24}n^4 + \frac{7}{12}n^6 - \frac{1}{2}n^7 + \frac{1}{8}n^8$$

$$\sum_{i=1}^{n-1} i^8 = -\frac{1}{30}n + \frac{2}{9}n^3 - \frac{7}{15}n^4 + \frac{2}{3}n^6 - \frac{1}{2}n^8 + \frac{1}{9}n^9$$

$$\sum_{i=1}^{n} i^9 = -\frac{3}{20}n^2 + \frac{1}{2}n^4 - \frac{7}{10}n^6 + \frac{3}{4}n^8 - \frac{1}{2}n^9 + \frac{1}{10}n^{10}$$

$$\sum_{i=1}^{n} i^{10} = \frac{5}{66}n - \frac{1}{2}n^3 + n^5 - n^7 + \frac{5}{6}n^9 - \frac{1}{2}n^{10} + \frac{1}{11}n^{11}$$

Note that these polynomials differ from those originally considered by Faulhaber and Bernoulli only for the sign of the coefficient of the monomy having a degree less than that of the polynomial belonging.

4. Theorem 2B. It is proved as a corollary of the previous theorem 1A, a formula for obtaining Bernoulli numbers from the Pascal's triangle

We have seen that the matrix A_n so defined

$$a_{j,k} = \begin{cases} 0 & \text{if } k > j \\ \binom{j}{k-1} & \text{otherwise} \end{cases}$$

has an inverse matrix that has the first-degree coefficients of polynomials in the first column. These correspond to "first Bernoulli numbers" where $B_1 = -½$ stands. The "second Bernoulli numbers" differs only for $B_1 = +½$.

Recalling the Laplace algorithm for the calculation of the inverse matrices we have;

$$c_{n,1} = B_{n-1} = (-1)^{n+1} \frac{|A_{1,n}|}{|A_n|} = (-1)^{n+1} \frac{|A_{1,n}|}{n!}$$

where:

$c_{n,1}$ indicates the corresponding element of $a_{n,1}$ in the reverse matrix

$| A_n | = n!$ Is the determinant of the triangular matrix.

$A_{1,n}$ is the algebraic complement (obtained by deleting first row and last column) relative to the corresponding $a_{1,n}$ element of $a_{n,1}$ in the transposition for the inverse calculation.

It is therefore possible to define Bernoulli numbers as follows:

$$\boxed{B_n = (-1)^n \frac{|A_n|}{(n+1)!}}$$

Where however $| A_n |$ is the determinant of a n order matrix defined as follows:

$$\boxed{a_{i,k} = \begin{cases} 0 & \text{if } k > 1 + i \\ \binom{i+1}{k-1} & \text{otherwise} \end{cases}}$$

This formula gives first Bernoulli numbers. If the factor $(-1)^n$, is omitted, since the only Bernoulli number from zero with odd index is B_1, we get the second Bernoulli numbers:

$$\boxed{B_n = \frac{|A_n|}{(n+1)!}}$$

Examples:

$$B_1 = \frac{\begin{vmatrix} 1 \end{vmatrix}}{2!} = \frac{1}{2}$$

$$B_2 = \frac{\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{3!} = \frac{1}{6}$$

$$B_3 = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 1 & 3 & 3 \\ 1 & 4 & 6 \end{vmatrix}}{4!} = \frac{0}{24} = 0$$

$$B_4 = \frac{\begin{vmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix}}{5!} = \frac{-4}{120} = -\frac{1}{30}$$

$$B_5 = \frac{\begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 \\ 1 & 5 & 10 & 10 & 5 \\ 1 & 6 & 15 & 20 & 15 \end{vmatrix}}{6!} = \frac{0}{720} = 0$$

$$B_6 = \frac{\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \\ 1 & 7 & 21 & 35 & 35 & 21 \end{vmatrix}}{7!} = \frac{120}{5040} = \frac{1}{42}$$

References

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