Pascal's identity and its modern demonstration

+ by Giorgio Pietrocola (from a letter to Nigel Derby 2018, August 19th)

In 1634 Pascal recursively solved the problem of searching for polynomials expressing the sums of powers of successive integers by this identity[1]:

$$(n+1)^r = (n+1) + r\sum_{k=1}^{n} k + \binom{r}{2}\sum_{k=1}^{n} k^2 + \binom{r}{3}\sum_{k=1}^{n} k^3 +,,, + \binom{r}{r-1}\sum_{k=1}^{r-1} k^{r-1}$$

Using the vector V (V as Vandermonde) of the increasing powers I also found a simple way to prove the formula discovered by Pascal[2].

$$\vec{V}(k) = \begin{pmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix}$$

$$\vec{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} k^0 \\ \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \end{pmatrix} = \sum_{k=1}^{4} \vec{V}(k)$$

Considering the development of a binomial, this generalizable identity is obvious:

[1] Mcmillan Sondow, Proof of power sum and binomial coefficient congruences via Pascal's identity, Internet Archive, 2010
Mathematical association of America Janet Beery Sums of Powers of Positive Integers - Blaise Pascal (1623-1662)

[2] A.W.F. Edwards Pascal treatise in Pascals Arhitmetical Triangle pp.82-85
Mcmillan, Sondow Proof of power sum and binomial coefficient congruences via Pascal's identity digitalizzato su Internet archive, 2010

$$\begin{pmatrix} (n+1)^1 - n \\ (n+1)^2 - n^2 \\ (n+1)^3 - n^3 \\ (n+1)^4 - n^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 4 & 6 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ n^1 \\ n^2 \\ n^3 \end{pmatrix}$$

$$(n+1)\vec{V}(n+1) - n\vec{V}(n) = P\vec{V}(n)$$

Where P is the matrix of Pascal’s triangle without the last element. With this vector identity it is very easy to demonstrate Pascal's formula after having expressed it with the language of matrices:

$$(n+1)^r = (n+1) + r\sum_{k=1}^{n} k + \binom{r}{2}\sum_{k=1}^{n} k^2 + \binom{r}{3}\sum_{k=1}^{n} k^3 +,,, + \binom{r}{r-1}\sum_{k=1}^{r-1} k^{r-1}$$

$$(n+1)^r - 1 = \binom{r}{0}\sum_{k=1}^{n} k^0 + \binom{r}{1}\sum_{k=1}^{n} k^1 + \binom{r}{2}\sum_{k=1}^{n} k^2 + \binom{r}{3}\sum_{k=1}^{n} k^3 +,,, + \binom{r}{r-1}\sum_{k=1}^{r-1} k^{r-1}$$

$$(n+1)^1 - 1 = \binom{1}{0}\sum_{k=1}^{n} k^0$$

$$(n+1)^2 - 1 = \binom{2}{0}\sum_{k=1}^{n} k^0 + \binom{2}{1}\sum_{k=1}^{n} k^1$$

$$(n+1)^3 - 1 = \binom{3}{0}\sum_{k=1}^{n} k^0 + \binom{3}{1}\sum_{k=1}^{n} k^1 + \binom{3}{2}\sum_{k=1}^{n} k^2$$

$$(n+1)^4 - 1 = \binom{4}{0}\sum_{k=1}^{n} k^0 + \binom{4}{1}\sum_{k=1}^{n} k^1 + \binom{4}{2}\sum_{k=1}^{n} k^2 + \binom{4}{3}\sum_{k=1}^{n} k^3$$

$$\begin{pmatrix} (n+1)^1 - 1 \\ (n+1)^2 - 1 \\ (n+1)^3 - 1 \\ (n+1)^4 - 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 4 & 6 & 4 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

$$(n+1)\vec{V}(n+1) - \vec{V}(1) = P\vec{S}$$

Indeed

$$P\vec{S} = P\sum_{k=1}^{4} \vec{V}(k) = \sum_{k=1}^{4} P\vec{V}(k) = P\vec{V}(1) + P\vec{V}(2) + P\vec{V}(3) + P\vec{V}(4)$$

$$= 2\vec{V}(2) - \vec{V}(1) + 3\vec{V}(3) - 2\vec{V}(2) + 4\vec{V}(4) - 3\vec{V}(3) + 5\vec{V}(5) - 4\vec{V}(4) = (n+1)\vec{V}(n+1) - \vec{V}(1)$$

where n=4 but generalization is easy

(IT) **Nota su un’ identità attribuita a Pascal in** Sui polinomi per il calcolo di somme di potenze di interi successivi e sui numeri di Bernoulli dedotti e dimostrati con il triangolo di Tartaglia