From binomial matrices to various theorems on the extension of Faulhaber's formula, on Bernoulli numbers and polynomials

Giorgio Pietrocola
giorgio.pietrocola[at]gmail.com
www.pietrocola.eu

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Abstract

Starting from a few elements, binomial matrices and vectors, without resorting to the traditional expansion in power series, polynomials and Bernoulli numbers are defined, proving various theorems on them. Along the way, the classic problem of the sum of powers of successive integers, which is generalized to sums of powers having any arithmetic progression as a basis, is also solved, in various ways, . The itinerary develops through numerous propositions rigorously linked on the classic model of Euclid's Elements. Numerous examples and hypertext references have the didactic purpose of making reading easier.

Keywords— Binomial matrices, Bernoulli polynomials, Bernoulli numbers, Faulhaber's formulas, linear algebra

Contents

1 Introduction

1.1 Warnings

These will be binomial matrices, i.e. matrices of order m, potentially infinite, which follow in whole or in part the development of the power of the binomial and therefore Pascal's triangle. The integer indices r and c will be used to indicate the number of rows and columns of the various matrices taken into consideration. The variability of these from 1 to m will often be implied. In order not to burden the notation, we have also chosen not to make explicit the order m of the matrices in the formulas, which can be, from time to time, fixed as desired in the examples. Any vectors present will

be considered as matrices with a single column and m rows.
We will set $0^0 = 1$, this will allow in several cases to generalize without making exceptions when the non-negative integer exponent vanishes. Example: $\sum_{k=0}^{n-1} k^0 = n$

2 Definitions

2.1 Three diagonal matrices: N, J, U

Definition 1 (N) *Diagonal matrix of natural numbers*

$$[N]_{r,c} = r \quad \textit{if } c = r, \qquad \textit{otherwise } 0$$

It is applied in: E:1,2; P:6,7,15,16,18,19,27,29,30,31,51,53,54,55,56, 57,60,61.

Definition 2 (U J, matrix group : unit and its root)

$$[U]_{r,c} = 1 \quad \textit{if } c = r, \qquad \textit{otherwise } 0$$

$$[J]_{r,c} = (-1)^{r-1} \quad \textit{if } c = r, \qquad \textit{otherwise } 0$$

It is applied in: E:1 P:6,8,16,18,19,21,22,23,24,25,27,29,30,31,34,38, 51,53,54,55,56,57,59.

Notes U J N are diagonal matrices ie square matrices of order m with all null elements except those of the main diagonal. The product rows by columns between matrices of this type boils down to the multiplication of elements that is, corresponding to the Hadamard product ($XY = X \circ Y$). These matrices form a multiplicative group. The inverse matrix of a diagonal matrix therefore has the elements of the diagonal corresponding to each other. U and J they are inverses of themselves.

2.2 Three costant Pascal matrices: T, A e Z

By the term binomial matrix we mean matrices of order m constructed using the binomial coefficients following Pascal's triangle

Definition 3 (T matrix) *Full Pascal triangle*

$$[T]_{r,c} = \binom{r-1}{c-1} \quad \textit{if } c \le r, \qquad \textit{otherwise } 0$$

It is applied in: E:5; D:15; P:1,2,3,6,7,8,9,10,11,14,15,16,17,18,19,26,27, 28,29,30,31,39,40,42,54,57,61.

Definition 4 (A matrix) *Pascal triangle without last element of row*

$$[A]_{r,c} = \binom{r}{c-1} \quad \textit{if } c \le r, \qquad \textit{otherwise } 0$$

It is applied in: D:7; P:15,16,18,19,28,31,32,33,34,35,36,37,38,39,40, 41,43,44,46.

Definition 5 (Z matrix) *Pascal triangle without first element of row*

$$[Z]_{r,c} = \binom{r}{c} \quad \textit{if } c \le r, \qquad \textit{otherwise } 0$$

It is applied in: E:5 P:6,7,8,27,28,30,53,55,56.

2.3 Two variable binomial matrices: T(h,d), G(h,d)

Definition 6 ***T(h,d)*** *Non Abelian group of binomial matrices of order m*

$$[T(h,d)]_{r,c} = \binom{r-1}{c-1} h^{r-c} d^{c-1} \quad \text{if } c \le r, \quad \text{otherwise } 0 \qquad h,d \in \mathbb{C}$$

Special cases
$T(h,1) = T^h$ *group of the powers of* T *(P:5)*
$T(1,1) = T^1 = T$ *full Pascal triangle*
$T(0,d) = \tilde{V}(d)$ *group of diagonal Vandermonde matrices (P:4)*
$T(0,-1) = \tilde{V}(-1) = J$ *unit root*
$T(0,1) = T^0 = \tilde{V}(1)$ *unit*

It is applied in: D7; P:4,5,9,11,13,14,29,39; E:9.

Definition 7 ***G(h,d) matrices***

$$G(h,d) = T(h,d)A^{-1}$$

Special cases:
$G(h,1) = T^h A^{-1}$
$G(0,1) = G_0 = A^{-1}$
$G(1,1) = G_1 = TA^{-1}$

It applies: D:4,15; is applied in: D:7; P:9,29,30,31,33,39,40,41,42,43,44, 46,47,48,49,53,54,55,56,57,60.

2.4 Three variable vectors: V(x), B(x), S(h,d,x)

Definition 8 ***Vandermonde vector*** *Let* $\vec{V}(j)$ *be the vectors with* m *components defined as follow:*

$$[\vec{V}(x)]_r = V_{r-1} = x^{r-1} \text{ with } r = 1,2,3,...m \quad r,m \in \mathbb{N}^+ \quad x \in \mathbb{C}$$

It is applied in: D:9,10; E:2 P:3,4,5,6,9,10,11,13,15,16,17,18,20,28,29,30, 31,32,33,34,35,36,37,38,39,43,44,45,46,53,54,55,56,57,59,60,61.

Definition 9 (Bernoulli vector)

$$\vec{B}(x) = A^{-1}N\vec{V}(x) \qquad x \in \mathbb{C}$$

Special cases:
$\vec{B} = \vec{B}(0)$
$\vec{B}^+ = \vec{B}(1)$
It applies: D:8; is applied in: E:6; P:15,16,17,18,19,29,30,31,47, 49,50,51, 52,53,54,55,56,57,58,59,60,61,62.

Note In P:15 we will show that the components $B_k(x)$ of the vector thus defined are the Bernoulli polynomials and that the sequences $B_k = B_k(0)$ and $B_k^+ = B_k(1)$ are the two variants of the Bernoulli numbers

Definition 10 ***S(h,d,x) sum powers vector*** *with* m *components:* $x,h,d \in \mathbb{C}$

$$\vec{S}(h,d,x) = G(h,d)x\vec{V}(x)$$

Special cases:
$\vec{S}(h,1,x) = G(h,1)x\vec{V}(x) = T^h A^{-1} x\vec{V}(x)$
$\vec{S}(0,1,x) = \vec{S}(x) = G(0,1)x\vec{V}(x) = G_0 x\vec{V}(x) = A^{-1}x\vec{V}(x)$
$\vec{S}(1,1,x) = \vec{S}^+(x) = G(1,1)x\vec{V}(x) = G_1 x\vec{V}(x) = TA^{-1}x\vec{V}(x)$
It applies: D:7,8; is applied in: P:33,37,39,44,45,46,53,54,55,56,57,60,61.

2.5 Three operators: semi-opposite, tilde and hat

Definition 11 (Semi-opposite operator) *From matrices to similar matrices with alternating signs This operator trasforms a matrix X in another similar one $\overline{X}$, called semi-opposite of X, with elements $[X]_{r,c}$ opposite in the position where $r+c$ id odd, i.e. sach that:*

$$[\overline{X}]_{r,c} = (-1)^{r+c}[X]_{r,c}$$

It is applied in: E:3,4; P:21,22,23,24,25,26,36,37,38,40,41,42,48,59.

Note It is easy to see, Si constata facilmente, as better explained in P:21 and E:3, that

$$\overline{X} = JXJ$$

Definition 12 (Tilde operator) *From vector to diagonal matrices*

Withy reference to a vector $\vec{X}$ with components $X_0, X_1, ...X_{m-1}$ gives a diagonal matrix defined as follows:

$$[\tilde{X}]_{r,c} = X_r \quad \textit{if } c = r \quad \textit{otherwise 0}$$

It applies: D:8; is applied in: E:1 P:3,4,7,8,11,13,14,16,21,24,25,27, 29,30, 31,34,38,39,53,54,55,56,57,60.

Recall that the Hadamard product (symbol $\circ$) between matrices of the same order is a matrix with elements equal to the product of the corresponding elements.

Definition 13 (Hat operator) *From vectors to Toeplitz triangular matrices With reference to a vector $\vec{X}$ with components $X_0, X_1, ...X_{m-1}$ it gives a triangular Toeplitz matrix defined as follows:*

$$[\widehat{X}]_{r,c} = X_{r-c} \quad \textit{if } c \leq r \quad \textit{otherwise 0}$$

It is applied in: E:8 D:15,16 P:1,2,3,5,6,7,17,18,19,20,27,29,30, 53,54,55,56,57,58,60,61.

2.6 T-composed matrices

Definition 14 (T-composed matrices) *We will call T -composed matrices having structure $T \circ \widehat{X}$*
Recall that the Hadamard product (symbol $\circ$) between matrices of the same order is a matrix with elements equal to the product of the corresponding elements. Similarly matrices of the type $Z \circ \widehat{X}$ will be called Z-composed

It applies: D:3,5,13; is applied in: E:9 P:1,2,3,5,7,16,19,20,27,28,29,30, 31,53,54,55,56,57,60,61.

2.7 links to three other definitions

Further definitions:

D:15 Powers of T

D:16 Powers of Z

D:17 Binomial matrix for Bernoulli numbers

2.8 Collection of examples on the given definitions

Example 1 (Diagonal matrices) *m=6*

$$N=\begin{bmatrix}1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&3&0&0&0\\0&0&0&4&0&0\\0&0&0&0&5&0\\0&0&0&0&0&6\end{bmatrix}\quad N^{-1}=\begin{bmatrix}1&0&0&0&0&0\\0&\frac{1}{2}&0&0&0&0\\0&0&\frac{1}{3}&0&0&0\\0&0&0&\frac{1}{4}&0&0\\0&0&0&0&\frac{1}{5}&0\\0&0&0&0&0&\frac{1}{6}\end{bmatrix}\quad N\begin{bmatrix}x\\x^2\\x^3\\x^4\\x^5\\x^6\end{bmatrix}=\begin{bmatrix}1\\2x\\3x^2\\4x^3\\5x^4\\6x^5\end{bmatrix}$$

$=$

$$\tilde{V}(2)=\begin{bmatrix}1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&4&0&0&0\\0&0&0&8&0&0\\0&0&0&0&16&0\\0&0&0&0&0&32\end{bmatrix}\quad U=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\quad J=\begin{bmatrix}1&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\end{bmatrix}$$

It applies: D: 1,2,12,8.

Example 2 (Vandermonde vectors) *with m=6 components*

$$\vec{V}(j)=\begin{bmatrix}1\\j\\j^2\\j^3\\j^4\\j^5\end{bmatrix}=\begin{bmatrix}V_0\\V_1\\V_2\\V_3\\V_4\\V_5\end{bmatrix}\qquad j\vec{V}(j)=\begin{bmatrix}j\\j^2\\j^3\\j^4\\j^5\\j^6\end{bmatrix}\qquad \vec{V}(0)=\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x\vec{V}(x)=\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix}x\\x^2\\x^3\\x^4\\x^5\\x^6\end{bmatrix}=\begin{bmatrix}1\\2x\\3x^2\\4x^3\\5x^4\\6x^5\end{bmatrix}=N\vec{V}(x)=\begin{bmatrix}1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&3&0&0&0\\0&0&0&4&0&0\\0&0&0&0&5&0\\0&0&0&0&0&6\end{bmatrix}\begin{bmatrix}1\\x\\x^2\\x^3\\x^4\\x^5\end{bmatrix}$$

It applies: D: 1,8; is applied in: P: 60.

Example 3 (T, J matrices) *with m=4*

$$JT=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&-1&0&0\\1&2&1&0\\-1&-3&-3&-1\end{bmatrix}$$

$$TJ=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\1&-1&0&0\\1&-2&1&0\\1&-3&3&-1\end{bmatrix}$$

$$JTJ=\begin{bmatrix}1&0&0&0\\-1&-1&0&0\\1&2&1&0\\-1&-3&-3&-1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\-1&1&0&0\\1&-2&1&0\\-1&3&-3&1\end{bmatrix}$$

Note that the multiplication of J on the left causes the rows to change sign alternately while on the right the columns alternately

It applies: *D: 3,11 P:21;*

Example 4 (Example of semi-opposite matrices) *m=6*

$$A=\begin{bmatrix}1&0&0&0&0&0\\1&2&0&0&0&0\\1&3&3&0&0&0\\1&4&6&4&0&0\\1&5&10&10&5&0\\1&6&15&20&15&6\end{bmatrix}\qquad \overline{A}=\begin{bmatrix}1&0&0&0&0&0\\-1&2&0&0&0&0\\1&-3&3&0&0&0\\-1&4&-6&4&0&0\\1&-5&10&-10&5&0\\-1&6&-15&20&-15&6\end{bmatrix}$$

It applies: D: 11

Example 5 (T, Z matrices) *with m=5*

$$T=\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&2&1&0&0\\1&3&3&1&0\\1&4&6&4&1\end{bmatrix}\quad \overline{T}=\begin{bmatrix}1&0&0&0&0\\-1&1&0&0&0\\1&-2&1&0&0\\-1&3&-3&1&0\\1&-4&6&-4&1\end{bmatrix}\quad Z=\begin{bmatrix}1&0&0&0&0\\2&1&0&0&0\\3&3&1&0&0\\4&6&4&1&0\\5&10&10&5&1\end{bmatrix}$$

It applies: *D: 3,4,5*

Example 6 (Bernoulli vectors) *m=6 components*

$$\vec{B}(x)=\begin{bmatrix}B_0(x)\\B_1(x)\\B_2(x)\\B_3(x)\\B_4(x)\\B_5(x)\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&0\\1&2&0&0&0&0\\1&3&3&0&0&0\\1&4&6&4&0&0\\1&5&10&10&5&0\\1&6&15&20&15&6\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&3&0&0&0\\0&0&0&4&0&0\\0&0&0&0&5&0\\0&0&0&0&0&6\end{bmatrix}\begin{bmatrix}1\\x\\x^2\\x^3\\x^4\\x^5\end{bmatrix}=$$

$$=\begin{bmatrix}1&0&0&0&0&0\\-\frac{1}{2}&\frac{1}{2}&0&0&0&0\\\frac{1}{6}&-\frac{1}{2}&\frac{1}{3}&0&0&0\\0&\frac{1}{4}&-\frac{1}{2}&\frac{1}{4}&0&0\\-\frac{1}{30}&0&\frac{1}{3}&-\frac{1}{2}&\frac{1}{5}&0\\0&-\frac{1}{12}&0&\frac{5}{12}&-\frac{1}{2}&\frac{1}{6}\end{bmatrix}\begin{bmatrix}1\\2x\\3x^2\\4x^3\\5x^4\\6x^5\end{bmatrix}=\begin{bmatrix}1\\-\frac{1}{2}+x\\\frac{1}{6}-x+x^2\\-\frac{3}{2}+x^2+x^3\\-\frac{1}{30}+x^2-2x^3+x^4\\-\frac{1}{6}x+\frac{5}{3}x^3-\frac{5}{2}x^4+x^5\end{bmatrix}$$

or for the associative property

$$=\begin{bmatrix}1&0&0&0&0&0\\-\frac{1}{2}&1&0&0&0&0\\\frac{1}{6}&-1&1&0&0&0\\0&\frac{1}{2}&-\frac{3}{2}&1&0&0\\-\frac{1}{30}&0&1&-2&1&0\\0&-\frac{1}{6}&0&\frac{5}{3}&-\frac{5}{2}&1\end{bmatrix}\begin{bmatrix}1\\x\\x^2\\x^3\\x^4\\x^5\end{bmatrix}=\begin{bmatrix}1\\-\frac{1}{2}+x\\\frac{1}{6}-x+x^2\\-\frac{1}{2}x-\frac{3}{2}x^2+x^3\\-\frac{1}{30}+x^2-2x^3+x^4\\-\frac{1}{6}x+\frac{5}{3}x^3-\frac{5}{2}x^4+x^5\end{bmatrix}$$

$$\vec{B}(0)=\vec{B}=\begin{bmatrix}B_0\\B_1\\B_2\\B_3\\B_4\\B_5\end{bmatrix}=\begin{bmatrix}1\\-\frac{1}{2}\\\frac{1}{6}\\0\\-\frac{1}{30}\\0\end{bmatrix}\qquad \vec{B}(1)=\vec{B}^{+}=\begin{bmatrix}B_0^+\\B_1^+\\B_2^+\\B_3^+\\B_4^+\\B_5^+\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{2}\\\frac{1}{6}\\0\\-\frac{1}{30}\\0\end{bmatrix}$$

It applies: D:9

Example 7 (Binomial matrices) *m=5*

$$T(h,d)=\begin{bmatrix}1&0&0&0&0\\h&d&0&0&0\\h^2&2hd&d^2&0&0\\h^3&3h^2d&3hd^2&d^3&0\\h^4&4h^3d&6h^2d^2&4hd^3&d^4\end{bmatrix}\quad T(1,1)=T=\begin{bmatrix}1&0&0&0&0\\1&1&0&0&0\\1&2&1&0&0\\1&3&3&1&0\\1&4&6&4&1\end{bmatrix}$$

$$T(h,1)=T^h=\begin{bmatrix}1&0&0&0&0\\h&1&0&0&0\\h^2&2h&1&0&0\\h^3&3h^2&3h&1&0\\h^4&4h^3&6h^2&4h&1\end{bmatrix}\quad T(0,d)=\tilde{V}(d)=\begin{bmatrix}1&0&0&0&0\\0&d&0&0&0\\0&0&d^2&0&0\\0&0&0&d^3&0\\0&0&0&0&d^4\end{bmatrix}$$

It applies: D:6 It is applied n: E:12,14

Example 8 (Toeplitz matrices) *$m=6$ matrices that have in the first column $\vec{V}(h)$ or $\vec{B}(h)$ vectors*

$$\widehat{V}(h)=\begin{bmatrix}V_0(h)&0&0&0&0&0\\V_1(h)&V_0(h)&0&0&0&0\\V_2(h)&V_1(h)&V_0(h)&0&0&0\\V_3(h)&V_2(h)&V_1(h)&V_0(n)&0&0\\V_4(h)&V_3(h)&V_2(h)&V_1(h)&V_0(h)&0\\V_5(h)&V_4(h)&V_3(h)&V_2(h)&V_1(h)&V_0(h)\end{bmatrix}$$

$$\widehat{B}(h)=\begin{bmatrix}B_0(h)&0&0&0&0&0\\B_1(h)&B_0(h)&0&0&0&0\\B_2(h)&B_1(h)&B_0(h)&0&0&0\\B_3(h)&B_2(h)&B_1(h)&B_0(n)&0&0\\B_4(h)&B_3(h)&B_2(h)&B_1(h)&B_0(h)&0\\B_5(h)&B_4(h)&B_3(h)&B_2(h)&B_1(h)&B_0(h)\end{bmatrix}$$

It applies: D:9;13

Example 9 (T matrices) *$T(h,1)$ matrices of the pèowers of T and of Z. The first is $T(h,1)=T^h=$:*

$$=\begin{bmatrix}1&0&0&0&0&0\\h&1&0&0&0&0\\h^2&2h&1&0&0&0\\h^3&3h^2&3h&1&0&0\\h^4&4h^3&6h^2&4h&1&0\\h^5&5h^4&10h^3&10h^2&5h&1\end{bmatrix}\quad Z^h=\begin{bmatrix}1&0&0&0&0&0\\2h&1&0&0&0&0\\3h^2&3h&1&0&0&0\\4h^3&6h^2&4h&1&0&0\\5h^4&10h^3&10h^2&5h&1&0\\6h^5&15h^4&20h^3&15h^2&6h&1\end{bmatrix}$$

Note that the first is a T-composite matrix, the second a Z-composite one
It applies: D:14,15,16;

Example 10 (T-composite matrix and Hadamard product) *m=6*

$$T\circ\widehat{B}(h)=\begin{bmatrix}1B_0(h)&0&0&0&0&0\\1B_1(h)&1B_0(h)&0&0&0&0\\1B_2(h)&2B_1(h)&1B_0(h)&0&0&0\\1B_3(h)&3B_2(h)&3B_1(h)&1B_0(h)&0&0\\1B_4(h)&4B_3(h)&6B_2(h)&4B_1(h)&1B_0(h)&0\\1B_5(h)&5B_4(h)&10B_3(h)&10B_2(h)&5B_1(h)&1B_0(h)\end{bmatrix}$$

$$Z \circ \widehat{B}(h) = \begin{bmatrix} 1B_0(h) & 0 & 0 & 0 & 0 & 0 \\ 2B_1(h) & 1B_0(h) & 0 & 0 & 0 & 0 \\ 3B_2(h) & 3B_1(h) & 1B_0(h) & 0 & 0 & 0 \\ 4B_3(h) & 6B_2(h) & 4B_1(h) & 1B_0(h) & 0 & 0 \\ 5B_4(h) & 10B_3(h) & 10B_2(h) & 5B_1(h) & 1B_0(h) & 0 \\ 6B_5(h) & 15B_4(h) & 20B_3(h) & 15B_2(h) & 6B_1(h) & 1B_0(h) \end{bmatrix}$$

$$T \circ \widehat{V}(h) = \begin{bmatrix} 1V_0(h) & 0 & 0 & 0 & 0 & 0 \\ 1V_1(h) & 1V_0(h) & 0 & 0 & 0 & 0 \\ 1V_2(h) & 2V_1(h) & 1V_0(h) & 0 & 0 & 0 \\ 1V_3(h) & 3V_2(h) & 3V_1(h) & 1V_0(n) & 0 & 0 \\ 1V_4(h) & 4V_3(h) & 6V_2(h) & 4V_1(h) & 1V_0(h) & 0 \\ 1V_5(h) & 5V_4(h) & 10V_3(h) & 10V_2(h) & 5V_1(h) & 1V_0(h) \end{bmatrix}$$

Note that the first and third are T-composite matrices, while the second is Z-composite
It applies: D: 14

3 From the umbral theorem to the translation of Bernoulli polynomials

3.1 Umbral theorem and commutativity of T-matrices

Proposition 1 (Umbral theorem) *Indices as exponents of powers of binome. Given two matrices of order m, T-composed by the Hadamard product between T and a triangular Toeplitz matrix, their row times column product turns out such that:* [1]*:*

$$(T \circ \widehat{X})(T \circ \widehat{Y}) = T \circ \widehat{R} \quad \text{where } \vec{R} \text{ has components} \quad R_j = \sum_{k=0}^{j} \binom{j}{k} X_{j-k} Y_k$$

for j ranging from 0 to $m-1$
It applies: D:3,13,14 is applied: E:11; P:2,3,17

Indeed keeping in mind D:3 and D:13 the components of $T \circ \widehat{R}$ are:

$$[T \circ \widehat{R}]_{r,c} = \binom{r-1}{c-1} \sum_{k=0}^{r-c} \binom{r-c}{k} X_{r-c-k} Y_k \quad \text{if } c \leq r, \text{ otherwise } 0$$

Also starting from the components:

$$[T \circ \widehat{X}]_{r,k} = \binom{r-1}{k-1} X_{r-k} \quad \text{if } c \leq r, \text{ otherwise } 0$$

$$[T \circ \widehat{Y}]_{k,c} = \binom{k-1}{c-1} Y_{k-c} \quad \text{if } c \leq r, \text{ otherwise } 0$$

The row-by-column product of the two matrices excluding the zero summation values ($[T \circ \widehat{Y}]_{k,c}) = 0$ if $k < c$) due to triangolarity is:

$$\sum_{k=c}^{r} [T \circ \widehat{X}]_{r,k} [T \circ \widehat{Y}]_{k,c} = \sum_{k=c}^{r} \binom{r-1}{k-1} X_{r-k} \binom{k-1}{c-1} Y_{k-c} =$$

$$= \sum_{k=c}^{r} \frac{(r-1)!}{(k-1)!(r-k)!} \frac{(k-1)!}{(c-1)!(k-c)!} X_{r-k} Y_{k-c} =$$

[1]since the indices act as the exponents in the development of the power of the binomial the conventions of umbral calculus allow us to write $R_j = (X+Y)^j$

$$= \frac{(r-1)!}{(c-1)!(r-c)!} \sum_{k=c}^{r} \frac{(r-c)!}{(r-k)!(k-c)!} X_{r-k} Y_{k-c} = \binom{r-1}{c-1} \sum_{k=c}^{r} \binom{r-c}{k-c} X_{r-k} Y_{k-c} =$$

$$= \binom{r-1}{c-1} \sum_{k=0}^{r-c} \binom{r-c}{k} X_{r-c-k} Y_k = [T \circ \widehat{R}]_{r,c}$$

In fact, as can be verified, the variations in the last summation do not vary the addends. q.e.d.

Proposition 2 (Commutativity corollary of composite T-matrices)

$$(T \circ \widehat{X})(T \circ \widehat{Y}) = (T \circ \widehat{Y})(T \circ \widehat{X})$$

It applies: D:3,13,14 P:1 is applied in: E:11; P:16,18,28;

Immediate consequence of P:1 and of the commutativity of ordinary multiplication. q.e.d.

Example 11 (Umbral) *with m=4 components*

$$\begin{bmatrix} 1X_0 & 0 & 0 & 0 \\ 1X_1 & 1X_0 & 0 & 0 \\ 1X_2 & 2X_1 & 1X_0 & 0 \\ 1X_3 & 3X_2 & 3X_1 & 1X_0 \end{bmatrix} \begin{bmatrix} 1Y_0 & 0 & 0 & 0 \\ 1Y_1 & 1Y_0 & 0 & 0 \\ 1Y_2 & 2Y_1 & 1Y_0 & 0 \\ 1Y_3 & 3Y_2 & 3Y_1 & 1Y_0 \end{bmatrix} = \begin{bmatrix} 1R_0 & 0 & 0 & 0 \\ 1R_1 & 1R_0 & 0 & 0 \\ 1R_2 & 2R_1 & 1R_0 & 0 \\ 1R_3 & 3R_2 & 3R_1 & 1R_0 \end{bmatrix}$$

$R_0 = 1X_0Y_0$
$R_1 = 1X_1Y_0 + 1X_0Y_1$
$R_2 = 1X_2Y_0 + 2X_1Y_1 + 1X_0Y_2$
$R_3 = 1X_3Y_0 + 3X_2Y_1 + 3X_1Y_2 + 1X_0Y_3$
It applies: P: 1,2

3.2 Additivity and definition of powers of T

Proposition 3 (Additivity of matrices T composed with V)

$$(T \circ \widehat{V}(h))(T \circ \widehat{V}(q)) = T \circ \widehat{V}(h+q)$$

It applies: D:3,8,13,14; P:1 is applied in: P:7,8; D:15.

For umbral theorem (P:1) the product is $T \circ \widehat{R}$ with

$$R_j = \sum_{j=0}^{j} \binom{j}{k} V_k(h) V_{j-k}(q) = \sum_{j=0}^{j} \binom{j}{k} h^k q^{j-k} = (h+q)^j = V_j(h+q)$$

for which $\vec{R} = \vec{V}(h+q)$ and therefore the result of the product of the two T-composed matrices is $T \circ \widehat{V}(h+q)$. q.e.d.

Definition 15 (Matrices of powers of T)) T^h *(T-composed matrices)*

$$T^h = T \circ \widehat{V}(h) \qquad h \in \mathbb{C}$$

Special cases: $T^0 = U \quad T^1 = T$
It applies: D:3,8,13; P:3; is applied in: E:9 P:5,8,9,10,11,13,14,16,17, 18,20,26,27,28,29,30,31,39,42,60,61.

3.3 Two Special cases of T(h,d)

Proposition 4 (First identity for T(0,d))

$$T(0,d) = \tilde{V}(d)$$

It applies: D:6,8,12; is applied in: P:7,8,13,14.

Per D:6

$$[T(0,d)]_{r,c} = \binom{r-1}{c-1} 0^{r-c} d^{c-1} \quad \text{if } c \le r, \quad \text{otherwise } 0$$

For the initial convention of considering $0^0 = 1$ only the terms with $r = c$r do not they cancel. So the previous one becomes

$$[T(0,d)]_{r,c} = d^{c-1} \quad \text{if } c = r, \quad \text{otherwise } 0$$

which is the definition of a diagonal matrix $(1, d, d^2..)$ i.e. $(V_0(d), V_1(d), V_2(d)...)$ and therefore precisely with the components of the vector $\vec{V}(d)$ as elements of the diagonal.

Proposition 5 (Second identity for T(h,1))

$$T(h,1) = T^h$$

It applies: D:6,8,13,14,15; is applied in: P:7,8,13,14.

For D:15 $T^h = T \circ \widehat{V}(h)$ Being $[T]_{r,c} = \binom{r-1}{c-1}$ if $c \le r$, otherwise 0 e
$[\widehat{V}(h)] = h^{r-c}$ if $c \le r$, otherwise 0
multiplying element by element we have that $[T \circ \widehat{V}(h)]_{r,c} = \binom{r-1}{c-1} h^{r-c}$
which coincides with D:6 $[T(h,d)]_{r,c} = \binom{r-1}{c-1} h^{r-c} d^{c-1}$ if $c \le r$, otherwise 0 when d=1. For the transitivity of equality the thesis follows. q.e.d.

3.4 Addittivity and definition of powers of Z

Proposition 6 (Relations between T-composed and Z-composed matrices)

$$Z \circ \tilde{V}(k) = N(T \circ \tilde{V}(k))N^{-1}$$

It applies: D:1,3,5,8,13,14; is applied in: P:27.

Multiplying on the left by N^{-1} we obtain the equivalent equation which we will prove:

$$N^{-1}(Z \circ \tilde{V}(k)) = (T \circ \tilde{V}(k))N^{-1}$$

Expanding the row-by-column product of the matrices of the first equality and taking into account the zeros in the matrices we have:
$\sum_{j=m}^{r} [N^{-1}]_{r,j} [Z \circ \tilde{V}(k)]_{j,c} = \sum_{j=r}^{r} \frac{1}{j} \binom{j}{c} V_{j-c}(k) = \frac{1}{r} \binom{r}{c} V_{r-c}(k)$
$\sum_{j=1}^{m} [T \circ \tilde{V}(k)]_{r,j} [N^{-1}]_{j,c} = \sum_{j=r}^{r} \binom{r-1}{j-1} \frac{1}{j} V_{r-j}(k) = \binom{r-1}{c-1} \frac{1}{c} V_{r-c}(k) = \frac{1}{r} \binom{r}{c} V_{r-c}(k)$
in fact it results:

$$\frac{1}{r}\binom{r}{c} = \frac{1}{r}\frac{r!}{c!(r-c)!} = \frac{1}{c}\frac{(r-1)!}{(c-1)!(r-c)!} = \binom{r-1}{j-1}\frac{1}{c}$$

q.e.d.

Proposition 7 (Addittivity of Z-composed matrices with V)

$$(Z \circ \widehat{V}(h))(Z \circ \widehat{V}(q)) = Z \circ \widehat{V}(h+q)$$

It applies: D:1,2,3,5,13,14; P:3,6; is applied in: P:8; D:16.

Substituting according to P:6 we obtain:

$$N(T \circ \widehat{V}(h))N^{-1}N(T \circ \widehat{V}(q))N^{-1} = N(T \circ \widehat{V}(h+q))N^{-1}$$

being $N^{-1}N = U$ we have:

$$N(T \circ \widehat{V}(h))(T \circ \widehat{V}(q))N^{-1} = N(T \circ \widehat{V}(h+q))N^{-1}$$

So the addition of T (P:3) proves equivalence and thus the thesis. q.e.d.

Definition 16 (Matrices of powers of Z)

$$Z^h = Z \circ V(h) \qquad h \in \mathbb{C}$$

Special cases: $Z^0 = U \quad Z^1 = Z$
It applies: D:2,5,8,13; P:7; is applied in: E:9,10 P:8,27,28,30,53

3.5 Abelian groups and no

Proposition 8 (Abelian groups of powers)

$$T^h \quad \text{additive group of powers of } T$$

$$Z^h \quad \text{additive group of powers of } Z$$

$$\tilde{V}(p) \quad \text{Vandermondian multiplicative group}$$

It applies: D:2,3,5,12,15,16; P:3,7; is applied in: P:9;

The set of matrices T^h with respect to the product of matrices forms a abelian group isomorphic to ordinary numerical sum Indeed, for P:3 and for the associative property we have:
$T^1 = T \quad T^hT^{-h} = T^0 = U \quad T^hT^q = T^qT^h = T^{h+q}$
Similarly for P:7 we have:
$Z^1 = Z \quad Z^hZ^{-h} = Z^0 = U \quad Z^hZ^q = Z^qZ^h = Z^{h+q}$
For $p \neq 0$ the product rows by columns between diagonal matrices gives
$\tilde{V}(1) = U \quad \tilde{V}(p)\tilde{V}(\frac{1}{p}) = U \quad \tilde{V}(p)\tilde{V}(q) = \tilde{V}(pq)$
q.e.d.

Proposition 9 ***Matrices for linear trasformation***

$$T(h,d)\vec{V}(j) = \vec{V}(h+dj)$$

Particular case: (traslation of $\vec{V}(j)$*)*

$$T^h\vec{V}(j) = \vec{V}(h+j)$$

It applies: D:6,8,15; is applied in: E:12; P:10,16,18,39.

Carrying out the product rows by columns $\sum_{k=1}^{m}[T(h,d)]_{r,k}[\vec{V}(j)]_k$:
substituting according to the definitions given and remembering that, for triangularity of the matrix, $\sum_{k=1}^{r}\binom{r-1}{k-1}h^{r-k}d^{k-1}j^{k-1} = \sum_{k=1}^{r}\binom{r-1}{k-1}h^{r-k}(dj)^{k-1} = (h+dj)^{r-1} = [\vec{V}(h+dj)]_r$ q.e.d.

Example 12 (Binomial matrices) $m = 6 \quad T(h,d)$ *we have:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ h & d & 0 & 0 & 0 & 0 \\ h^2 & 2hd & d^2 & 0 & 0 & 0 \\ h^3 & 3h^2d & 3hd^2 & d^3 & 0 & 0 \\ h^4 & 4h^3d & 6h^2d^2 & 4hd^3 & d^4 & 0 \\ h^5 & 5h^4d & 10h^3d^2 & 10h^2d^3 & 5hd^4 & p^5 \end{bmatrix} \begin{bmatrix} 1 \\ j \\ j^2 \\ j^3 \\ j^4 \\ j^5 \end{bmatrix} = \begin{bmatrix} 1 \\ h+dj \\ (h+dj)^2 \\ (h+dj)^3 \\ (h+dj)^4 \\ (h+dj)^5 \end{bmatrix}$$

expressible as:

$$T(h,d)\vec{V}(j) = \vec{V}(h+dj)$$

It applies: P:9

Proposition 10 (Non commutative group) *The set of matrices $T(h,d)$ with respect to the product it forms a non-commutative group isomorphic to that of composition of linear functions with one variable*
It applies: D:3,8,15,6; P:9; is applied in: P:11.

Per P:9 si ha $T(h,d)\vec{V}(j) = \vec{V}(h+dj)$. $T(h,d)$ matrix induces a linear transformation on the variable of the Vandermonde vector by which it multiplies. The product of two matrices of this type, by the associativity of the product rows by columns, corresponds to the composition of two linear functions q.e.d.

3.6 Properties of T(h,d)

Proposition 11 (Product of linear terasformations)

$$T(a,b)T(x,y) = T(a+bx, by)$$

It applies: D:3,6,8,15; P:10; is applied in: E:13; P:13,14.

In fact, the result is $T(a,b)T(x,y)\vec{V}(j) = T(a,b)\vec{V}(x+yj) =$
$= \vec{V}(a+b(x+yj)) = \vec{V}(a+bx+byj) = T(a+bx,by)\vec{V}(j) =$
$= T(a+bx,by)\vec{V}(j)$ from which, comparing the extremes, the thesis. q.e.d.

Proposition 12 (Inverse of a linear transformation) *For $b \neq 0$*

$$T(h,d)T(-\frac{h}{d},\frac{1}{d}) = T(-\frac{h}{d},\frac{1}{d})T(h,d) = T(0,1) = U$$

It applies: D:3,6,8,15; P:11; is applied in: P:13,14.

Immediate consequence of the product rule seen previously (P:11) q.e.d.

Example 13 (Inverse binomial matrices) $m=4,\ T(-\frac{h}{d},\frac{1}{d})^{-1}T(h,d) = U$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{h}{d^2} & \frac{1}{d} & 0 & 0 \\ \frac{h^2}{d^4} & -2\frac{h}{d^3} & \frac{1}{d^2} & 0 \\ -\frac{h^3}{d^6} & 3\frac{h^2}{d^5} & -3\frac{h}{d^4} & 1\frac{1}{d^3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ h & d & 0 & 0 \\ h^2 & 2hd & d^2 & 0 \\ h^3 & 3h^2d & 3hd^2 & d^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It applies: P:12.

Proposition 13 (Decomposition of T(h,d) into the product of two subgroups)

$$T(h,d) = T^h\tilde{V}(d)$$

It applies: D:6,8,12,15; P:4,5,11; is applied in: E:14; P:14,29,42.

Recalling that for P:4,5 it result $T(h,1) = T^h$ e $T(0,d) = \tilde{V}(d)$ and calculating the product $T(1,h)T(0,d)$ on the basis of P:11 gives the thesis. q.e.d.

Proposition 14 (Decomposed Product Inversion)

$$T^h\tilde{V}(d) = \tilde{V}(d)T^{\frac{h}{d}}$$

It applies: D:3,6,8,12,15; P:4,5,11,13; is applied in: E:14,15; P:29,42.

For P:11 risulta che $T(0,d)T(0,\frac{h}{d}) = T(h,d)$. Substituting according to P:4,5,13 you get the thesis. q.e.d.

Example 14 (Decomposed binomial matrices) $m = 4 \quad T^h\widehat{V}(d) = T(h,d):$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ h & 1 & 0 & 0 \\ h^2 & 2h & 1 & 0 \\ h^3 & 3h^2 & 3h & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h & d & 0 & 0 \\ h^2 & 2hd & d^2 & 0 \\ h^3 & 3h^2d & 3hd^2 & d^3 \end{bmatrix}$$

It applies: P:13.

Example 15 (Decomposed binomial matrices) $m = 4 \quad \tilde{V}(d)T^{\frac{h}{d}} = T(h,d):$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{h}{d} & 1 & 0 & 0 \\ \frac{h^2}{d^2} & 2\frac{h}{d} & 1 & 0 \\ \frac{h^3}{d^3} & 3\frac{h^2}{d^2} & 3\frac{h}{d} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ h & d & 0 & 0 \\ h^2 & 2hd & d^2 & 0 \\ h^3 & 3h^2d & 3hd^2 & d^3 \end{bmatrix}$$

It applies: P:14.

3.7 Inversion and translations in Bernoulli Polynomials

Proposition 15 (Inversion property of Bernoulli polynomials) *Provides recursive formulas to express the Bernoulli polynomials in terms of the previous ones.*

$$A\vec{B}(x) = N\vec{V}(x)$$

m-th row for column: $\sum_{k=1}^{m} \binom{m}{k-1} B_{k-1}(x) = mx^{m-1}$

Special case: $\quad A\vec{B} = \vec{V}(0)$

m-th row for column: $\sum_{k=1}^{m} \binom{m}{k-1} B_{k-1} = \begin{cases} 1 & \textit{if } m = 1 \\ 0 & \textit{if } m > 1 \end{cases}$

It applies: D:1,3,4,8,9; is applied in: P:62.

the first equation is obtained from the definition $\vec{B}(n) = A^{-1}N\vec{V}(n)$ (D:9) by multiplying the two sides on the left by A.
Passing to the m-th component we obtain the well-known inversion formula of Bernoulli polynomials which allows us to obtain them recursively. q.e.d.

The particular case can be explained with the simple observation that $N\vec{V}(0) = \vec{V}(0)$ By varying m this equation, solved with respect to the index Bernoulli number major, provides a simple recursive formula for calculating these numbers. It was precisely this formula that in 1842 was chosen to be implemented in that first program that anticipated the information age by over a century.[11]

Proposition 16 (Translation Vector B)

$$T^h\vec{B}(n) = \vec{B}(h+n)$$

m-th row by column:

$$\sum_{k=1}^{m} \binom{m-1}{k-1} B_{m-1-k}(n)n^k = B_m(h+n)$$

It applies: D:1,3,4,8,9,12,14,15; P:2,9; is applied in: P:17,18,20,58.

D:9 give us $\vec{B}(n) = A^{-1}N\vec{V}(n)$ from which inversely, multiplying the two left-sided for $N^{-1}A$, we obtein $\vec{V}(n) = N^{-1}A\vec{B}(n)$ by substituting this value in the equation $T^h\vec{V}(n) = \vec{V}(n+h)$, obteined from P:9 considering $T^h = T(h,1)$, we obtein:

$$T^hN^{-1}A\vec{B}(n) = N^{-1}A\vec{B}(h+n)$$

multiplying on the left by $A^{-1}N$

$$A^{-1}NT^hN^{-1}A\vec{B}(n) = N^{-1}A\vec{B}(h+n)$$

Recalling D:15 for witch $T^h = T\circ\vec{V}(h)$, the power of T is a T-composed matrix and, for P:2, commutative, therefore $A^{-1}NT^h = T^hA^{-1}N$. Finally, being $A^{-1}NN^{-1}A = U$ the thesis follows. q.e.d.

Note The latter can be expressed with the umbral calculus convention even so:

$$\big(q + B(h)\big)^{m-1} = B(q+h)$$

Proposition 17 (Translations for T-composed Bernoulli matrices)

$$T^q(T\circ\widehat{B}(h)) = T\circ\widehat{B}(q+h)$$

Special case:
$T^q(T\circ\widehat{B}) = T\circ\widehat{B}(q)$
It applies: D:3,8,9,13,15; P:1,16; is applied in: P:20,29.

For D:15 it result $T^h = T\circ\widehat{V}(h)$ and therefore by the umbral theorem (P:1) the product between T-composed matrices must be a T-composed matrix of the form T $T\circ\widehat{R}$ with

$$R_j = \sum_{k=0}^{j}\binom{j}{k}V_k(h)B_{j-k}(q) = \sum_{k=0}^{j}\binom{j}{k}h^kB_{j-k}(q) =$$
$$= \sum_{k=1}^{j+1}\binom{j}{k-1}h^{k-1}B_{j-k+1}(q) = B_{j+1}(h+q)$$

in fact the last two summations as k varies differently produce the same effects on indices, exponents and binomial coefficients and therefore by virtue of the previous P:16 we herefore have $\vec{R} = \vec{B}(h+q)$ and therefore the matrix $\widehat{B}(h+q)$ q.e.d.

Proposition 18 (Faulhaberian bridge theorem) *Establishes a new bridge between $\vec{V}(h)$ and $\vec{B}(h)$ after D:9*

$$\vec{B}(h) = (T\circ\widehat{B})\vec{V}(h)$$

m-th row by column:

$$\sum_{k=1}^{m}\binom{m-1}{k-1}B_{m-k}n^{k-1} = B_{m-1}(h)$$

It applies: D:8,12,13,15; P:2,9,16; is applied in: E:16; P:19,20.

The thesis is true for h=0:

$$\vec{B}(0) = (T\circ\widehat{B})\vec{V}(0)$$

In fact the vector $\vec{V}(0)$, first column of U matrix, gives us the first column of matrix $T\circ\widehat{B}$ which is the element-by-element product between $\vec{V}(1)$ first column of T and $\vec{B}$ first column of $\widehat{B}$
From this particular equation, multiplying the two sides on the left by T^h we obtain:

$$T^h\vec{B}(0) = T^h(T\circ\widehat{B})\vec{V}(0)$$

By the traslation theorem $\vec{B}$ (P:16) and being $T^h = T \circ \vec{V}(h)$ for the commutativity of T-composed matrices (P:2)

$$\vec{B}(h) = (T \circ \widehat{B})T^h\vec{V}(0)$$

finally, $T^h\vec{V}(0) = \vec{V}(h)$ by the translation theorem (P:9) then follows the thesis. q.e.d.

Proposition 19 ($A^{-1}N$ is a T-composed matrix) *Faulhaberian corollary.*

$$A^{-1}N = T \circ \widehat{B}$$

It applies: D:1,3,4,9,13; P:18,28; is applied in: E:16; P:29.

From the comparison of D:9
$\vec{B}(h) = A^{-1}N\vec{V}(h)$
and P:18
$\vec{B}(h) = (T \circ \widehat{B})\vec{V}(h)$
the thesis is deduced. q.e.d.

Example 16 ***order m=4*** $\quad (T \circ \widehat{B})\vec{V}(n) =$

$$\left(\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix} \circ \begin{bmatrix}B_0&0&0&0\\B_1&B_0&0&0\\B_2&B_1&B_0&0\\B_3&B_2&B_1&B_0\end{bmatrix}\right)\begin{bmatrix}1\\n\\n^2\\n^3\end{bmatrix} =$$

$$= \begin{bmatrix}\binom{0}{0}B_0&0&0&0\\\binom{1}{0}B_1&\binom{1}{1}B_0&0&0\\\binom{2}{0}B_2&\binom{2}{1}B_1&\binom{2}{2}B_0&0\\\binom{3}{0}B_3&\binom{3}{1}B_2&\binom{3}{2}B_1&\binom{3}{3}B_0\end{bmatrix}\begin{bmatrix}1\\n\\n^2\\n^3\end{bmatrix} =$$

$$= \begin{bmatrix}\binom{0}{0}B_0\\\binom{1}{0}B_1 + \binom{1}{1}B_0 n\\\binom{2}{0}B_2 + \binom{2}{1}B_1 n + \binom{2}{2}B_0 n^2\\\binom{3}{0}B_3 + \binom{3}{1}B_2 n + \binom{3}{2}B_1 n^2 + \binom{3}{3}B_0 n^3\end{bmatrix} = \begin{bmatrix}\sum_{k=1}^{1}\binom{0}{k-1}B_{1-k}n^{k-1}\\\sum_{k=1}^{2}\binom{1}{k-1}B_{2-k}n^{k-1}\\\sum_{k=1}^{3}\binom{2}{k-1}B_{3-k}n^{k-1}\\\sum_{k=1}^{4}\binom{3}{k-1}B_{4-k}n^{k-1}\end{bmatrix} =$$

$$= \begin{bmatrix}B_0(n)\\B_1(n)\\B_2(n)\\B_3(n)\end{bmatrix} = \begin{bmatrix}1\\-\frac{1}{2}+n\\\frac{1}{6}-n+n^2\\\frac{1}{2}n-\frac{3}{2}n^2+n^3\end{bmatrix}$$

is applied in: P:19,20.

Proposition 20 (Generalized bridge theorem)

$$(T \circ \widehat{B}(q))\vec{V}(h) = \vec{B}(h+q)$$

m-th row per column:

$$\sum_{k=1}^{m}\binom{m-1}{k-1}B_{m-k}(q)n^{k-1} = B_{m-1}(h+q)$$

It applies: D:8,13,14,15; P:16,17,18; is applied in: E:16; P:60.

Indeed for the P:18 we have:

$$(T \circ \widehat{B}(0))\vec{V}(h) = \vec{B}(h)$$

Moreover per P:17 we have:
$T^q(T \circ \widehat{B}(0)) = T \circ \widehat{B}(q)$
and for P:16 si ha:
$T^q\vec{B}(h) = \vec{B}(h+q)$
So multiplying the two sides on the left by T^q the thesis follows. q.e.d.

Example 17 (Matrix $T\circ\vec{B}(h)$ and Bernoulli polynomials) *order m=6)* $(T\circ \widehat{B}(h))\vec{V}(n) =$

$$= \begin{bmatrix} 1B_0(h) & 0 & 0 & 0 & 0 & 0 \\ 1B_1(h) & 1B_0(h) & 0 & 0 & 0 & 0 \\ 1B_2(h) & 2B_1(h) & 1B_0(h) & 0 & 0 & 0 \\ 1B_3(h) & 3B_2(h) & 3B_1(h) & 1B_0(h) & 0 & 0 \\ 1B_4(h) & 4B_3(h) & 6B_2(h) & 4B_1(h) & 1B_0(h) & 0 \\ 1B_5(h) & 5B_4(h) & 10B_3(h) & 10B_2(h) & 5B_1(h) & 1B_0(h) \end{bmatrix} \begin{bmatrix} 1 \\ n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \end{bmatrix} = \vec{B}(n+h)$$

is applied in: P:20.

4 Properties of matrices in semi-opposite relationship

Proposition 21 (semi-opposed matrices) *Double multiplication by J*

$$\overline{X} = JXJ$$

It applies: D:2,11; It is applied in: E:3; P:22,23,24,25,26.

In fact, setting $[X]_{r,c} = x_{r,c}$ and remembering thet diagonal matrix
$[J]_{r,c} = (-1)^{r+1}$ if r=c otherwise 0
multiplying rows by columns JX changes sign to even rows for which $[JX]_{r,c} = x_{r,c}(-1)^{r+1}$ Subsequent multiplication changes sign to even columns so $[JXJ]_{r,c} = x_{r,c}(-1)^{r+1}(-1)^{c+1} = x_{r,c}(-1)^{r+c+2} = x_{r,c}(-1)^{r+c}$ So X and JXJ are in semi-opposition relations for which $\overline{X} = JXJ$. q.e.d.

4.1 Properties

Proposition 22 (Double semi-opposition) *Double moltiplication by J The semi-opposite of the semi-opposite of X is X itself:*

$$\overline{\overline{X}} = X$$

It applies: D:2,11; P:21.

Indeed $J(JXJ)J = UXU = X$ q.e.d.

Proposition 23 (Simmetric property) *Semi-opposition reciprocity*

$$Y = \overline{X} \quad \textit{equals} \quad X = \overline{Y}$$

It applies: D:2,11; P:21.

Indeed $Y = \overline{X}$ for P:21 can be written in the form $Y = JXJ$. Multiplying the two members on the left by J we obtain the equivalent $JY = XJ$. Multiplying to the right, the X is made explicit obtaining $JYJ = X$ which corresponds to $X = \overline{Y}$ q.e.d.

Proposition 24 (Inverse matrices keep the semi-opposite relationship) *Given an invertible matrix X*

$$X = \overline{Y} \quad\iff\quad X^{-1} = \overline{Y^{-1}}$$

It applies: D:2,11; P:21; is applied in: P:41.

For P:21 the double implication is equivalent to $Y = JXJ \iff Y^{-1} = JX^{-1}J$
it is now easy to verify, remembering that $JJ = U$, that also the second members of the two equalities are inverse $JXJJX^{-1}J = JXX^{-1}J = JJ = U$ q.e.d.

Proposition 25 (The product maintains the semi-opposition)

$$X_1 = \overline{Y_1} \;\; and \;\; X_2 = \overline{Y_2} \qquad \Longleftrightarrow \qquad X_1X_2 = \overline{Y_1Y_2}$$

It applies: D:2,11; P:21; is applied in: P:42.

For P:21 the double implication is equivalent to
$Y_1 = JX_1J$ and $Y_2 = JX_2J \Longleftrightarrow X_1X_2 = JY_1Y_2J$
is now easy, remembering that $JJ = U$, verify that $Y_1Y_2 = JY_1JJY_2J = JY_1Y_2J$ q.e.d.

4.2 Relationship between the power of T and its inverse

Proposition 26 (Power of T and its inverse are semi-opposite) .

$$T^{-h} = \overline{T^h}$$

It applies: D:3,6,11,15; is applied in: P:42.

Per la D:15 $T^h = T \circ \widehat{V}(h)$ and $T^{-h} = T \circ \widehat{V}(-h)$.
For D:11 results:
$[\widehat{V}(h)]_{r,c} = V_{r-c}(h) = (h)^{r-c}$ and
$[\widehat{V}(-h)]_{r,c} = V_{r-c}(-h) = (-h)^{r-c} = (-1)^{r-c}h^{r-c}$.
Since 2c is even $(-1)^{2c} = 1$ then $(-1)^{r-c} = (-1)^{r-c}(-1)^{2c} = (-1)^{r+c}$ therefore when $r + c$ is even the components of the two Toeplitz matrices $\widehat{V}(h)$ and $\widehat{V}(-h)$ coincide and when $r + c$ is odd are opposite. So by definition (11) the two matrices are semi-opposite:

$$\widehat{V}(-h) = \overline{\widehat{V}(h)}$$

Moving on to the composite T-matrices, the Hadamard product must be performed by multiplying both, element by element, by the same T-matrix. Equal elements multiplied by the same number remain equal. Thus the opposite elements for which:

$$T \circ \widehat{V}(-h) = \overline{T \circ \widehat{V}(h)}$$

So by D:15 we get the thesis. q.e.d.

5 Notable product

5.1 Relations between matrices A,T and Z

Proposition 27 (Z-T relation)

$$Z^k = NT^kN^{-1}$$

It applies: D:1,3,5,8,13,14,15,16; P:6; is applied in: P:28,30.

Since $Z^h = Z \circ \widehat{V}(h)$ (D:15) It is an immediate consequence of P:6

Proposition 28 (T-A-Z relation)

$$T^hA^{-1} = A^{-1}Z^h$$

It applies: D:2,3,4,5,15,16; P:2,19,27.

Explaining A^{-1} in P:19 and substituting we have

$$T^h A^{-1} = T^h (T \circ \widehat{B}) N^{-1} =$$

for the commutativity of T-composite matrices (P:2)

$$= (T \circ \widehat{B}) T^h N^{-1} =$$

for the relation between the powers of T and of Z (P:27)

$$= (T \circ \widehat{B}) N^{-1} Z^h = A^{-1} Z^h$$

5.2 New expressions for G(h,d)

Proposition 29 (First alternative expression for G(h,d))

$$G(h,d) = \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}$$

Special cases:
$G(h,1) = (T \circ \widehat{B}(h))N^{-1}$
$G(1,1) = (T \circ \widehat{B}(1))N^{-1} = (T \circ \widehat{B}^{+})N^{-1}$
$G(0,1) = (T \circ \widehat{B}(0))N^{-1} = (T \circ \widehat{B})N^{-1}$
It applies: D:1,2,3,7,8,9,12,13,15; P:13,14,17,19; is applied in: E:18; P:54,61.

By the given definition (D:7) and for the equality $T(h,d) = T^h \tilde{V}(d) = \tilde{V}(d) T^{\frac{h}{d}}$ (P:13,14), being $NN^{-1} = U$, the result is $G(h,d) = \tilde{V}(d) T^{\frac{h}{d}} A^{-1} N N^{-1}$ substituting $A^{-1}N$ on the basis P:19 we obtein:

$$G(h,d) = \tilde{V}(d) T^{\frac{h}{d}} (T \circ \vec{B}) N^{-1}$$

Making the argument of $\vec{B}$ explicit and then applying the translation (P:17) the result is $T^{\frac{h}{d}}(T \circ \vec{B}(0)) = T \circ \vec{B}(\frac{h}{d})$ substituting finally we have the thesis. q.e.d.

Example 18 *order m=6* $\tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1} =$

$$G(h,d) = \begin{bmatrix} 1\frac{1}{1}B_0(\frac{h}{d}) & 0 & 0 & 0 & 0 & 0 \\ 1\frac{d}{1}B_1(\frac{h}{d}) & 1\frac{d}{2}B_0(\frac{h}{d}) & 0 & 0 & 0 & 0 \\ 1\frac{d^2}{1}B_2(\frac{h}{d}) & 2\frac{d^2}{2}B_1(\frac{h}{d}) & 1\frac{d^2}{3}B_0(\frac{h}{d}) & 0 & 0 & 0 \\ 1\frac{d^3}{1}B_3(\frac{h}{d}) & 3\frac{d^3}{2}B_2(\frac{h}{d}) & 3\frac{d^3}{3}B_1(\frac{h}{d}) & 1\frac{d^3}{4}B_0(\frac{h}{d}) & 0 & 0 \\ 1\frac{d^4}{1}B_4(\frac{h}{d}) & 4\frac{d^4}{2}B_3(\frac{h}{d}) & 6\frac{d^4}{3}B_2(\frac{h}{d}) & 4\frac{d^4}{4}B_1(\frac{h}{d}) & 1\frac{d^4}{5}B_0(\frac{h}{d}) & 0 \\ 1\frac{d^5}{1}B_5(\frac{h}{d}) & 5\frac{d^5}{2}B_4(\frac{h}{d}) & 10\frac{d^5}{3}B_3(\frac{h}{d}) & 5\frac{d^5}{4}B_2(\frac{h}{d}) & 6\frac{d^5}{5}B_1(\frac{h}{d}) & 1\frac{d^5}{6}B_0(\frac{h}{d}) \end{bmatrix}$$

It applies: P:29.

Proposition 30 (G expressed in Faulhaberian way)

$$G(h,d) = \tilde{V}(d) N^{-1} (Z \circ \widehat{B}(\frac{h}{d}))$$

$G(h,1) = N^{-1}(Z \circ \widehat{B}(h))$
$G(1,1) = N^{-1}(Z \circ \widehat{B}(1)) = N^{-1}(Z \circ \widehat{B}^{+})$
$G(0,1) = N^{-1}(Z \circ \widehat{B}(0)) = N^{-1}(Z \circ \widehat{B})$
It applies: D:1,2,3,5,7,8,9,12,13,15,16; P:27,29; is applied in: E:19 P:53.

We will show that the second member can transform into that of the previous one P:29. In fact, expressing Z as a function of T by P:27 and applying the properties of diagonal matrices with the Hadamard product we have:

$$\tilde{V}(d)N^{-1}(Z \circ \widehat{B}(\frac{h}{d})) = \tilde{V}(d)N^{-1}(NTN^{-1} \circ \widehat{B}(\frac{h}{d})) =$$

$$= \tilde{V}(d)N^{-1}N(TN^{-1} \circ \widehat{B}(\frac{h}{d})) = \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d})N^{-1}) = \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}$$

q.e.d.

Example 19 *order m=6* $\quad \tilde{V}(d)N^{-1}(Z \circ \widehat{B}(\frac{h}{d})) =$

$$G(h,d) = \begin{bmatrix} 1\frac{1}{1}B_0(\frac{h}{d}) & 0 & 0 & 0 & 0 & 0 \\ 2\frac{d}{2}B_1(\frac{h}{d}) & 1\frac{d}{2}B_0(\frac{h}{d}) & 0 & 0 & 0 & 0 \\ 3\frac{d^2}{3}B_2(\frac{h}{d}) & 3\frac{d^2}{3}B_1(\frac{h}{d}) & 1\frac{d^2}{3}B_0(\frac{h}{d}) & 0 & 0 & 0 \\ 4\frac{d^3}{4}B_3(\frac{h}{d}) & 6\frac{d^3}{4}B_2(\frac{h}{d}) & 4\frac{d^3}{4}B_1(\frac{h}{d}) & 1\frac{d^3}{4}B_0(\frac{h}{d}) & 0 & 0 \\ 5\frac{d^4}{5}B_4(\frac{h}{d}) & 10\frac{d^4}{5}B_3(\frac{h}{d}) & 10\frac{d^4}{5}B_2(\frac{h}{d}) & 5\frac{d^4}{5}B_1(\frac{h}{d}) & 1\frac{d^4}{5}B_0(\frac{h}{d}) & 0 \\ 6\frac{d^5}{6}B_5(\frac{h}{d}) & 15\frac{d^5}{6}B_4(\frac{h}{d}) & 20\frac{d^5}{6}B_3(\frac{h}{d}) & 15\frac{d^5}{6}B_2(\frac{h}{d}) & 6\frac{d^5}{6}B_1(\frac{h}{d}) & 1\frac{d^5}{6}B_0(\frac{h}{d}) \end{bmatrix}$$

It applies: P:30.

5.3 First column of G(h,1)

Proposition 31 (First column of G(h,1) matrices)

$$G(h,1)\vec{V}(0) = \vec{B}(h)$$

Special cases:

$$G(0,1)\vec{V}(0) = \vec{B}(0) = \vec{B} \quad e \quad G(1,1)\vec{V}(0) = \vec{B}(1) = \vec{B}^{+}$$

It applies: D:1,2,3,7,8,9,13,14,15; P:16,29;
is applied in: P:47,49,50,59.

For P:29 we have $G(0,1) = (T \circ \tilde{B})N^{-1}$ then the product $(T \circ \tilde{B})N^{-1}\vec{V}(0)$, $\vec{V}(0)$ being the first column of U, unit vector, gives the first column of $G(0,1)$.
It turn out $N^{-1}\vec{V}(0) = \vec{V}(0)$, first column of N^{-1},
$\tilde{B}\vec{V}(0) = \vec{B}$, first column of $\tilde{B}$
$T\vec{V}(0) = \vec{V}(1)$, first column of T. So

$$G(0,1)V\vec{(0)} = \vec{V}(1) \circ \vec{B} = \vec{B}$$

Now multiplying the two members of equality on the left by T^h applying D:7 and P:16 we get $G(h,1)V\vec{(0)} = \vec{B}(h)$ i.e. the thesis. q.e.d.

6 Sums of powers with arithmetic progression bases

6.1 G0 for successive integers starting from 0

6.1.1 G0 identity

Proposition 32 ***G0 identity***

$$A\vec{V}(k) = (1+k)\vec{V}(1+k) - k\vec{V}(k)$$

It applies: D:4,8; is applied in: E:20; P:33,35.
Taking into account that for the triangularity of the matrix the result is $[A]_{r,j} = 0$ if $j > r$, multiplying row by column, we have:

$$\sum_{j=1}^{m}[A]_{r,j}[\vec{V}(k)]_j = \sum_{j=1}^{r}\binom{r}{j-1}k^{j-1} = -k^r + \sum_{j=1}^{r+1}\binom{r}{j-1}k^{j-1} = (k+1)^r - k^r =$$

$$= [(k+1)\vec{V}(k+1)]_r - [k\vec{V}(k)]_r \qquad \text{q.e.d.}$$

Example 20 (Identity G0) *order m=6* $(1+k)\vec{V}(1+k) - k\vec{V}(k) =$

$$= \begin{bmatrix} (1+k)-k \\ (1+k)^2-k^2 \\ (1+k)^3-k^3 \\ (1+k)^4-k^4 \\ (1+k)^5-k^5 \\ (1+k)^6-k^6 \end{bmatrix} = \begin{bmatrix} (1+k) \\ (1+k)^2 \\ (1+k)^3 \\ (1+k)^4 \\ (1+k)^5 \\ (1+k)^6 \end{bmatrix} - \begin{bmatrix} k \\ k^2 \\ k^3 \\ k^4 \\ k^5 \\ k^6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+2k \\ 1+3k+3k^2 \\ 1+4k+6k^2+4k^3 \\ 1+5k+10k^2+10k^3+5k^4 \\ 1+6k+15k^2+20k^3+15k^4+6k^5 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \\ k^4 \\ k^5 \end{bmatrix} = A\vec{V}(k)$$

It applies: P:32.

6.1.2 G0 theorem

Proposition 33 (G0 theorem) *Solves the problem of adding powers of successive integers starting at 0.*

$$\vec{S}(n) = \sum_{k=0}^{n-1}\vec{V}(k)$$

It applies: D:4,7,8,10; P:32; is applied in: E:21; P:39,44.

Then adding, member by member, starting from 0, the first n Special cases of P:32 we obtain: $\sum_{k=0}^{n-1} A\vec{V}(k) = \sum_{k=0}^{n-1}\Big((1+k)\vec{V}(1+k) - k\vec{V}(k)\Big)$ Expanding the sum to the second member, almost all the terms are simplified (telescopic effect). Then collecting the matrix of the first member as a common factor and considering that $0\vec{V}(0) = 0$ we obtain: $A\sum_{k=0}^{n-1}\vec{V}(k) = n\vec{V}(n)$ from which by multiplying the two members on the left by the inverse of A (possible because the determinant is $m! \neq 0$), we obtain:

$$\sum_{k=0}^{n-1}\vec{V}(k) = A^{-1}n\vec{V}(n)$$

for D:7 we have $G_0 = A^{-1}$ for D:10 we have $\vec{S}(n) = G_0 n\vec{V}(n)$ from which, from which, for the transitivity of equality, the thesis follows. q.e.d.

Example 21 (G0 theorem) *Having chosen the case m=7 components and performing the product rows by columns:* $\vec{S}(n) = A^{-1}n\vec{V}(n)$

$$\vec{S}(n)=\begin{bmatrix}S_0(n)\\S_1(n)\\S_2(n)\\S_3(n)\\S_4(n)\\S_5(n)\\S_6(n)\end{bmatrix}=\begin{bmatrix}\sum_{k=0}^{n-1}k^0\\\sum_{k=0}^{n-1}k^1\\\sum_{k=0}^{n-1}k^2\\\sum_{k=0}^{n-1}k^3\\\sum_{k=0}^{n-1}k^4\\\sum_{k=0}^{n-1}k^5\\\sum_{k=0}^{n-1}k^6\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&0&0\\1&2&0&0&0&0&0\\1&3&3&0&0&0&0\\1&4&6&4&0&0&0\\1&5&10&10&5&0&0\\1&6&15&20&15&6&0\\1&7&21&35&35&21&7\end{bmatrix}^{-1}\begin{bmatrix}n\\n^2\\n^3\\n^4\\n^5\\n^6\\n^7\end{bmatrix}=$$

$$=\begin{bmatrix}1&0&0&0&0&0&0\\-\frac{1}{2}&\frac{1}{2}&0&0&0&0&0\\\frac{1}{6}&-\frac{1}{2}&\frac{1}{3}&0&0&0&0\\0&\frac{1}{4}&-\frac{1}{2}&\frac{1}{4}&0&0&0\\-\frac{1}{30}&0&-\frac{1}{3}&-\frac{1}{2}&\frac{1}{5}&0&0\\0&-\frac{1}{12}&0&\frac{5}{12}&-\frac{1}{2}&\frac{1}{6}&0\\\frac{1}{42}&0&-\frac{1}{6}&0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{7}\end{bmatrix}\begin{bmatrix}n\\n^2\\n^3\\n^4\\n^5\\n^6\\n^7\end{bmatrix}=\begin{bmatrix}n\\-\frac{1}{2}n+\frac{1}{2}n^2\\\frac{1}{6}n-\frac{1}{2}n^2+\frac{1}{3}n^3\\\frac{1}{4}n^2-\frac{1}{2}n^3+\frac{1}{4}n^4\\-\frac{1}{30}n+\frac{1}{3}n^3-\frac{1}{2}n^4+\frac{1}{5}n^5\\-\frac{1}{12}n^2+\frac{5}{12}n^4-\frac{1}{2}n^5+\frac{1}{6}n^6\\\frac{1}{42}n-\frac{1}{6}n^3+\frac{1}{2}n^5-\frac{1}{2}n^6+\frac{1}{7}n^7\end{bmatrix}$$

It applies: P:33

6.1.3 G0* theorem Similar to G0 but in factored form.

Proposition 34 (G0*, factored polynomials) *Similar to G0 but in factored form.*

$$\sum_{k=0}^{n-1}k\vec{V}(k)=(U+A)^{-1}(n-1)n\vec{V}(n)$$

It applies: D:2,4,8; E:22; P:32.

In fact, remembering that $A\vec{V}(k)=(1+k)\vec{V}(1+k)-k\vec{V}(k)$ we have:

$$(U+A)\sum_{k=0}^{n-1}k\vec{V}(k)=\sum_{k=0}^{n-1}(U+A)k\vec{V}(k)=$$

$$=\sum_{k=0}^{n-1}\Big(k\vec{V}(k)+kA\vec{V}(k)\Big)=\sum_{k=0}^{n-1}\Big(k\vec{V}(k)+k\Big((1+k)\vec{V}(1+k)-k\vec{V}(k)\Big)\Big)=$$

$$=\sum_{k=0}^{n-1}\Big(k\vec{V}(k)+k(k+1)\vec{V}(k+1)-k^2\vec{V}(k)\Big)=\sum_{k=0}^{n-1}\Big(k(k+1)\vec{V}(k+1)-k(k-1)\vec{V}(k)\Big)=$$

$=(n-1)n\vec{V}(n)$ In fact, after the substitution (P:32) all the terms of the summation due to the telescopic effect are simplified two by two except the first and the last. Considering the equality between the first and the last term of the identity chain and multiplying on the left the two members of the equality obtained by the inverse of $(U+A)$ with determinant $(m+1)!\neq 0$ we obtein $\sum_{k=0}^{n-1}k\vec{V}(k)=(U+A)^{-1}(n-1)n\vec{V}(n)$ q.e.d.

Example 22 (G0, factored polynomials) *order m=7*

$$\sum_{k=0}^{n-1}k\vec{V}(k)=\begin{bmatrix}\sum_{k=0}^{n-1}k^1\\\sum_{k=0}^{n-1}k^2\\\sum_{k=0}^{n-1}k^3\\\sum_{k=0}^{n-1}k^4\\\sum_{k=0}^{n-1}k^5\\\sum_{k=0}^{n-1}k^6\\\sum_{k=0}^{n-1}k^7\end{bmatrix}=\begin{bmatrix}2&0&0&0&0&0&0\\1&3&0&0&0&0&0\\1&3&4&0&0&0&0\\1&4&6&5&0&0&0\\1&5&10&10&6&0&0\\1&6&15&20&15&7&0\\1&7&21&35&35&21&8\end{bmatrix}^{-1}\begin{bmatrix}n(n-1)\\n^2(n-1)\\n^3(n-1)\\n^4(n-1)\\n^5(n-1)\\n^6(n-1)\\n^7(n-1)\end{bmatrix}=$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{30} & \frac{1}{30} & -\frac{3}{10} & \frac{1}{5} & 0 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{3} & \frac{1}{6} & 0 & 0 \\ -\frac{1}{42} & -\frac{1}{42} & \frac{1}{7} & \frac{1}{7} & -\frac{5}{14} & \frac{1}{7} & 0 \\ 0 & -\frac{1}{12} & -\frac{1}{12} & \frac{5}{24} & \frac{5}{24} & -\frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} n(n-1) \\ n^2(n-1) \\ n^3(n-1) \\ n^4(n-1) \\ n^5(n-1) \\ n^6(n-1) \\ n^7(n-1) \end{bmatrix} =$$

$$= \begin{bmatrix} n(n-1)\frac{1}{2} \\ n(n-1)(-\frac{1}{6}+\frac{1}{3}n) \\ n(n-1)(-\frac{1}{4}n+\frac{1}{4}n^2) \\ n(n-1)(\frac{1}{30}+\frac{1}{30}n-\frac{3}{10}n^2+\frac{1}{5}n^3) \\ n(n-1)(\frac{1}{12}n+\frac{1}{12}n^2-\frac{1}{3}n^3+\frac{1}{6}n^4) \\ n(n-1)(-\frac{1}{42}-\frac{1}{42}n+\frac{1}{7}n^2+\frac{1}{7}n^3-\frac{5}{14}n^4+\frac{1}{7}n^5) \\ n(n-1)(-\frac{1}{12}n-\frac{1}{12}n^2+\frac{5}{24}n^3+\frac{5}{24}n^4-\frac{3}{8}n^5+\frac{1}{8}n^6) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{n-1} k^1 \\ \sum_{k=0}^{n-1} k^2 \\ \sum_{k=0}^{n-1} k^3 \\ \sum_{k=0}^{n-1} k^4 \\ \sum_{k=0}^{n-1} k^5 \\ \sum_{k=0}^{n-1} k^6 \\ \sum_{k=0}^{n-1} k^7 \end{bmatrix}$$

It applies:P:34.

6.2 Pascal's identity

Proposition 35 (historical identity)

$$(n+1)\vec{V}(n+1) - \vec{V}(1) = A\sum_{k=1}^{n} \vec{V}(k)$$

The previous one expressed by the m-th component is equivalent to:

$$(n+1)^m - 1 = \binom{m}{0}\sum_{k=1}^{n} k^0 + \binom{m}{1}\sum_{k=1}^{n} k^1 + \binom{m}{2}\sum_{k=1}^{n} k^2 + \cdots + \binom{m}{m-1}\sum_{k=1}^{n} k^{m-1}$$

It applies: D:4,8; P:32; is applied in: S:10.

Vectorial equivalence is easily explained:

$$A\vec{S}_1(n) = A\sum_{k=1}^{n} \vec{V}(k) = \sum_{k=1}^{n} A\vec{V}(k) = \sum_{k=1}^{n}\Big((1+k)\vec{V}(1+k) - k\vec{V}(k)\Big) =$$

$= (n+1)\vec{V}(n+1) - \vec{V}(1)$ In fact, after the substitution (P:32) all the terms of the summation with effect telescopic simplify two by two except the first and last. q.e.d.

6.3 G1 per interi successivi inizianti da 1

6.3.1 G1 identity

Proposition 36 (G1 identity)

$$\overline{A}\vec{V}(k) = k\vec{V}(k) - (k-1)\vec{V}(k-1)$$

It applies: D:4,8,11; is applied in: E:23; P:37,38.

For the proof, we consider the product $\overline{A}\vec{V}(k)$ For triangularity of the matrix we get $[\overline{A}]_{r,j} = 0$ if $j > r$ therefore, multiplying row by column, we have:

$$\sum_{j=1}^{m} [\overline{A}]_{r,j}[\vec{V}(k)]_j = \sum_{j=1}^{r} \binom{r}{j-1}(-1)^{r+j}k^{j-1} = k^r + \sum_{j=1}^{r+1} \binom{r}{j-1}k^{j-1}(-1)^{r+j} =$$

$$= k^r - \sum_{j=1}^{r+1} \binom{r}{j-1} k^{j-1}(-1)^{r+j-1} = k^r - \sum_{q=0}^{r} \binom{r}{q} k^q (-1)^{r-q} =$$

(j has been replaced with $q+1$ and equivalently subtracted the equal $2q$ from the exponent of -1)
$= k^r - (k-1)^r = [k\vec{V}(k)]_r - [(k-1)\vec{V}(k-1)]_r$ q.e.d.

Example 23 (G1 identity) *order m=6*

$$\begin{bmatrix} k-(k-1) \\ k^2-(k-1)^2 \\ k^3-(k-1)^3 \\ k^4-(k-1)^4 \\ k^5-(k-1)^5 \\ k^6-(k-1)^6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1+2k \\ 1-3k+3k^2 \\ -1+4k-6k^2+4k^3 \\ 1-5k+10k^2-10k^3+5k^4 \\ -1+6k-15k^2+20k^3-15k^4+6k^5 \end{bmatrix}$$

Using vectors and matrices and developing the product rows by columns, the above becomes:

$$\begin{bmatrix} (k) \\ (k)^2 \\ (k)^3 \\ (k)^4 \\ (k)^5 \\ (k)^6 \end{bmatrix} - \begin{bmatrix} k-1 \\ (k-1)^2 \\ (k-1)^3 \\ (k-1)^4 \\ (k-1)^5 \\ (k-1)^6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ k \\ k^2 \\ k^3 \\ k^4 \\ k^5 \end{bmatrix}$$

which with the notation introduced simply corresponds to

$$k\vec{V}(k) - (k-1)\vec{V}(k-1) = \overline{A}\vec{V}(k)$$

It applies: P:36.

6.3.2 future G1 theorem

Proposition 37 (Future G1) *Solves the problem of the sum of powers of successive integers starting from 1 (even if D:7 still does not allow to deduce that $G_1 = \overline{A}$ and D:10 that $\vec{S}^+(n) = \sum_{k=0}^{n-1} \vec{V}(1+k)$).*

$$\sum_{k=0}^{n-1} \vec{V}(1+k) = \overline{A}^{-1} n\vec{V}(n)$$

It applies: D:4,8,10,11; P:36; is applied in: E:24; P:40,41.

Adding member by member, starting from 1, the first n Special cases of IG1 (P:36) we have:

$$\sum_{k=1}^{n} \overline{A}\vec{V}(k) = \sum_{k=1}^{n} \Big(k\vec{V}(k) - (k-1)\vec{V}(k-1) \Big)$$

Expanding the sum to the second member almost all the terms, except the first and the last, are simplified two by two with the opposites (telescopic effect). Taking into account that $-0\vec{V}(0) = 0$ and then collecting the matrix with the first member as a common factor, we obtain:

$$\overline{A} \sum_{k=1}^{n} \vec{V}(k) = n\vec{V}(n)$$

finally, to make the summation explicit, both sides of the equation on the left are multiplied by the inverse of the matrix $\overline{A}$ (existing because the triangular matrix has a determinant other than zero being the product of the diagonal equal to $m!$) obtaining

$$\sum_{k=1}^{n} \vec{V}(k) = \overline{A}^{-1} n\vec{V}(n) =$$

taking into account the equivalence of the two summation expressions which both indicate the n addends $\vec{V}(1), \vec{V}(2), ...\vec{V}(n)$ defining a vector whose components are the sums of the powers of successive integers starting from 1, the thesis follows. q.e.d

Example 24 (Future G1 theorem) *order m=11* $\vec{S}(1,1,n) = \overline{A}^{-1} n\vec{V}(n) =$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & -\frac{7}{24} & 0 & \frac{1}{12} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{2}{9} & 0 & -\frac{7}{15} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{9} & 0 & 0 \\ 0 & -\frac{3}{20} & 0 & \frac{1}{2} & 0 & -\frac{7}{10} & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{10} & 0 \\ \frac{5}{66} & 0 & -\frac{1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & \frac{1}{2} & \frac{1}{11} \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ n^9 \\ n^{10} \\ n^{11} \end{bmatrix} =$$

performing the product rows by columns:

$$= \begin{bmatrix} n \\ \frac{1}{2}n + \frac{1}{2}n^2 \\ \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 \\ \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4 \\ -\frac{1}{30}n + \frac{1}{3}n^3 + \frac{1}{2}n^4 + \frac{1}{5}n^5 \\ -\frac{1}{12}n^2 + \frac{5}{12}n^4 + \frac{1}{2}n^5 + \frac{1}{6}n^6 \\ \frac{1}{42}n - \frac{1}{6}n^3 + \frac{1}{2}n^5 + \frac{1}{2}n^6 + \frac{1}{7}n^7 \\ \frac{1}{12}n^2 - \frac{7}{24}n^4 + \frac{1}{12}n^6 + \frac{1}{2}n^7 + \frac{1}{8}n^8 \\ -\frac{1}{30}n + \frac{2}{9}n^3 - \frac{7}{15}n^5 + \frac{2}{3}n^7 + \frac{1}{2}n^8 + \frac{1}{9}n^9 \\ -\frac{3}{20}n^2 + \frac{1}{2}n^4 - \frac{7}{10}n^6 + \frac{3}{4}n^8 + \frac{1}{2}n^9 + \frac{1}{10}n^{10} \\ \frac{5}{66}n - \frac{1}{2}n^3 + n^5 - n^7 + \frac{5}{6}n^9 + \frac{1}{2}n^{10} + \frac{1}{11}n^{11} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n k^0 \\ \sum_{k=1}^n k^1 \\ \sum_{k=1}^n k^2 \\ \sum_{k=1}^n k^3 \\ \sum_{k=1}^n k^4 \\ \sum_{k=1}^n k^5 \\ \sum_{k=1}^n k^6 \\ \sum_{k=1}^n k^7 \\ \sum_{k=1}^n k^8 \\ \sum_{k=1}^n k^9 \\ \sum_{k=1}^n k^{10} \end{bmatrix}$$

It applies: P:37.

6.3.3 G1* theorem for factored polynomials

Proposition 38 (G1*, factored polynomials) *Similar to G1 but in factored form.*

$$\sum_{k=1}^{n} k\vec{V}(k) = (U + \overline{A})^{-1}(n+1)n\vec{V}(n)$$

It applies: D:2,4,8,11; P:36; is applied in: E:25.

$$(U + \overline{A})\sum_{k=1}^{n} k\vec{V}(k) = \sum_{k=1}^{n}(U + \overline{A})k\vec{V}(k) =$$

$$= \sum_{k=1}^{n}\Big(k\vec{V}(k) + k\overline{A}\vec{V}(k)\Big) = \sum_{k=1}^{n}\Big(k\vec{V}(k) + k\Big(k\vec{V}(k) - (k-1)\vec{V}(k-1)\Big)\Big) =$$

$$= \sum_{k=1}^{n}\Big(k\vec{V}(k) + k^2\vec{V}(k) - k(k-1)\vec{V}(k-1)\Big) = \sum_{k=1}^{n}\Big(k(1+k)\vec{V}(k) - k(k-1)\vec{V}(k-1)\Big) =$$

$= (n+1)n\vec{V}(n)$ In fact, after the substitution (P:36) all the terms of the summation due to the telescopic effect are simplified two by two except the first and the last. The thesis is obtained by considering the equality between the first and the last term of the identity chain and multiplying on the left the two members of the equality by the inverse of $(U + \overline{A})$ existing being the determinant $(m+1)! \neq 0$ q.e.d.

Example 25 (G1*, factored polynomials) *order m=7*

$$\sum_{k=1}^{n} k\vec{V}(k) = \begin{bmatrix} \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \\ \sum_{k=1}^{n} k^4 \\ \sum_{k=1}^{n} k^5 \\ \sum_{k=1}^{n} k^6 \\ \sum_{k=1}^{n} k^7 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 4 & 0 & 0 & 0 & 0 \\ -1 & 4 & -6 & 5 & 0 & 0 & 0 \\ 1 & -5 & 10 & -10 & 6 & 0 & 0 \\ -1 & 6 & -15 & 20 & -15 & 7 & 0 \\ 1 & -7 & 21 & -35 & 35 & -21 & 8 \end{bmatrix}^{-1} \begin{bmatrix} (n+1)n \\ (n+1)n^2 \\ (n+1)n^3 \\ (n+1)n^4 \\ (n+1)n^5 \\ (n+1)n^6 \\ (n+1)n^7 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{1}{30} & \frac{1}{30} & \frac{3}{10} & \frac{1}{5} & 0 & 0 & 0 \\ 0 & -\frac{1}{12} & \frac{1}{12} & \frac{1}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{42} & -\frac{1}{42} & -\frac{1}{7} & \frac{1}{7} & \frac{5}{14} & \frac{1}{7} & 0 \\ 0 & \frac{1}{12} & -\frac{1}{12} & -\frac{5}{24} & \frac{5}{24} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} n(n+1) \\ n^2(n+1) \\ n^3(n+1) \\ n^4(n+1) \\ n^5(n+1) \\ n^6(n+1) \\ n^7(n+1) \end{bmatrix} =$$

$$= \begin{bmatrix} n(n+1)\frac{1}{2} \\ n(n+1)(\frac{1}{6} + \frac{1}{3}n) \\ n(n+1)(\frac{1}{4}n + \frac{1}{4}n^2) \\ n(n+1)(-\frac{1}{30} + \frac{1}{30}n + \frac{3}{10}n^2 + \frac{1}{5}n^3) \\ n(n+1)(-\frac{1}{12}n + \frac{1}{12}n^2 + \frac{1}{3}n^3 + \frac{1}{6}n^4) \\ n(n+1)(\frac{1}{42} - \frac{1}{42}n - \frac{1}{7}n^2 + \frac{1}{7}n^3 + \frac{5}{14}n^4 + \frac{1}{7}n^5) \\ n(n+1)(\frac{1}{12}n - \frac{1}{12}n^2 - \frac{5}{24}n^3 + \frac{5}{24}n^4 + \frac{3}{8}n^5 + \frac{1}{8}n^6) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \\ \sum_{k=1}^{n} k^4 \\ \sum_{k=1}^{n} k^5 \\ \sum_{k=1}^{n} k^6 \\ \sum_{k=1}^{n} k^7 \end{bmatrix}$$

It applies: P:38.

6.4 Generalization to any arithmetic progression

6.5 Theorem G: for any arithmetic progression

Proposition 39 ***G theorem*** *It solves the general problem of finding the sum of powers of bases in arithmetic progression of which the traditional problem of the sum of powers of successive integers is a particular case.*

$$\vec{S}(h, d, n) = \sum_{k=0}^{n-1} \vec{V}(h + dk)$$

It applies: D:4,6,7,8,10; P:9,33; is applied in: P:40,53,54,61.

The statement of this proposition is obtained from $\vec{S}(n) = \sum_{k=0}^{n-1} \vec{V}(k)$ (P:33 G0 theorem) by multiplying the two members on the left by the matrix $T(h, d)$.
In fact, for D:10 and D:7 the result is $\vec{S}(n) = G_0 n\vec{V}(n) = A^{-1} n\vec{V}(n)$ and $G_(h, d) = T(h, d)A^{-1}$ so the first member of the equation to be multiplied becomes:

$$T(h, d)\vec{S}(n) = T(h, d)A^{-1} n\vec{V}(n) = G(h, d)n\vec{V}(n) = \vec{S}(h, d, n)$$

the second member by applying the distributive property and the linear transformation P:9:

$$T(h, d) \sum_{k=0}^{n-1} \vec{V}(k) = \sum_{k=0}^{n-1} T(h, d)\vec{V}(k) = \sum_{k=0}^{n-1} \vec{V}(h + dk)$$

q.e.d.

Example 26 (On the sums of Vandermonde vectors) *in arithmetic progression (with m=6 components)*

$$\vec{S}(0,1,n) = \sum_{k=0}^{n-1} \vec{V}(k) = \begin{bmatrix} \sum_{k=0}^{n-1} k^0 \\ \sum_{k=0}^{n-1} k^1 \\ \sum_{k=0}^{n-1} k^2 \\ \sum_{k=0}^{n-1} k^3 \\ \sum_{k=0}^{n-1} k^4 \\ \sum_{k=0}^{n-1} k^5 \end{bmatrix} \quad \vec{S}(1,1,n) = \sum_{k=0}^{n-1} \vec{V}(1+k) = \sum_{k=1}^{n} \vec{V}(k) = \begin{bmatrix} \sum_{k=1}^{n} k^0 \\ \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \\ \sum_{k=1}^{n} k^4 \\ \sum_{k=1}^{n} k^5 \end{bmatrix}$$

It applies: D: 10; P:39.

Example 27 ***G theorem : sums of powers of*** $3k+1$ ***progression.***
We now use the theorem G to compute sums of powers of integers which, when divided by three, give a remainder of 1. The result is $h = 1,\ d = 3$. *We choose* $m = 4$ *limiting ourselves to the first 4 polynomials. To apply the theorem G we have to consider:*

$$T(1,3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 6 & 9 & 0 \\ 1 & 9 & 27 & 27 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 4 & 6 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Having obtained the matrix $G(1,3) = T(1,3)A^{-1}$, *performing the row-by-column product, we have:*

$$\vec{S}(1,3,n) = \begin{bmatrix} S_0(1,3,n) \\ S_1(1,3,n) \\ S_2(1,3,n) \\ S_3(1,3,n) \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{n-1}(1+3k)^0 \\ \sum_{k=0}^{n-1}(1+3k)^1 \\ \sum_{k=0}^{n-1}(1+3k)^2 \\ \sum_{k=0}^{n-1}(1+3k)^3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{3}{2} & 3 & 0 \\ 1 & -\frac{9}{4} & -\frac{9}{2} & \frac{27}{4} \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \end{bmatrix} = \begin{bmatrix} n \\ -\frac{1}{2}n + \frac{3}{2}n^2 \\ -\frac{1}{2}n - \frac{3}{2}n^2 + 3n^3 \\ n - \frac{9}{4}n^2 - \frac{9}{2}n^3 + \frac{27}{4}n^4 \end{bmatrix}$$

It applies: P:39.

6.5.1 G1 theorem

Proposition 40 (G1 corollary) *It allows to express P:37 in the form* $\vec{S}^{+}(n) = G_1 n \vec{V}(n)$

$$G_1 = \overline{A}^{-1}$$

It applies: D:3,4,7,11; P:37,39.

From the definition D:7 it results $G_0 = A^{-1}$ for G theorem (P:39) as a special case $h = 1$, $d = 1$ we have
$\vec{S}^{+}(n) = \sum_{k=0}^{n-1} \vec{V}(1+k) = G_1 n \vec{V}(n)$ con $G_1 = TA^{-1}$ from P:37 we have $\sum_{k=0}^{n-1} \vec{V}(1+k) = \overline{A}^{-1} n \vec{V}(n)$ the comparison implies the thesis.

Proposition 41 (G0 and G1 are in semi-opposition relation)

$$G_0 = \overline{G_1}$$

It applies: D:4,7,11; P:23,24,40; is applied in: 42,47,48.

From definition D:7 it results $G_0 = A^{-1}$ from P:40 it results $G_1 = \overline{A}^{-1}$ therefore $G_0^{-1} = A$ is semi-opposite of $G_0^{-1} = \overline{A}$. Since A and $\overline{A}$ are in a semi-opposition relationship for P:24 so are G_0 and G_1. So it will result in $G_1 = \overline{G_0}$ and vice versa (P:23). q.e.d.

6.6 Semi-opposite pairs of matrices G

Proposition 42 (Semi-opposite pairs of G matrices)

$$G(h,d) = \overline{G}(d-h,d)$$

It applies: D:3,7,11,15; P:13,14,25,26,41; is applied in: P:59.

P:41 establishes that $G(0,1)$ and $G(1,1)$ which for brevity we denote by G_0 and G_1 are semi-opposite. For P:26 so are T^h and T^{-h}. Like all diagonal matrices $\tilde{V}(d)$ having all elements with odd row-column sum null, it is semi-opposite of itself. So for P:25 the products $T^h\tilde{V}(d)$ and $T^{-h}\tilde{V}(d)$ are also semi-opposite. For the same reason, the products must be semi-opposite:

$$T^h\tilde{V}(d)\; G_0 \quad \text{e} \quad T^{-h}\tilde{V}(d)\; G_1$$

The first product for P:13 and D:7 gives:

$$T(h,d)G_0 = G(h,d)$$

The second, given that $G_1 = TG_0$ for D:7, is:

$$T^{-h}\tilde{V}(d)\; TG_0$$

thus replacing $\tilde{V}(d)\; T$ with $T^d\tilde{V}(d)$ based on P:14 we have:

$$T^{-h}T^d\tilde{V}(d)\; TG_0 = T^{d-h}\tilde{V}(d) = G(d-h,d)$$

q.e.d.

7 Bernoulli numbers

Proposition 43 (Coefficients of the highest degree monomials) $G_0 = G(0,1) = A^{-1}$

$$\forall m \in \mathbb{N}^+ \quad [G_0]_{m,m} = \frac{1}{m}$$

It applies: D:4,7; is applied in: P:44

Bearing in mind the method of algebraic complements for calculating the inverse matrix:

$$[A^{-1}]_{m,m} = (-1)^{m+m}\frac{|A_{m,m}|}{|A_m|} = \frac{(m-1)!}{m!} = \frac{1}{m}$$

q.e.d

Proposition 44 (Polynomials of degree r) *A triangular matrix G_0 of order m contains, in each row r ($r = 1...m$) the coefficients of polynomials of degree r calculating sums of powers of successive integers starting from 0:*

$$[G_0]_{r,1}n + [G_0]_{r,2}n^2 + ... + [G_0]_{r,r}n^r = \sum_{k=0}^{n-1} k^{r-1}$$

It applies: D:4,7,8,10; P:33,43; is applied in: 46.

Indeed for G0 theorem (P:33) we have

$$\vec{S}(0,1,n) = G_0 n\vec{V}(n) = \sum_{k=0}^{n-1} \vec{V}(k)$$

taking into account that for the triangularity of the matrix $[G_0]_{r,j} = 0$ per $j > r$,the product of the row r for the vector $n\vec{V(n)}$ gives:

$$[G_0]_{r,1}n + [G_0]_{r,2}n^2 + ... + [G_0]_{r,r}n^r = \sum_{k=0}^{n-1} k^{r-1}$$

From P:43 we know that $[G_0]_{r,r} = \frac{1}{r} \neq 0$ so the degree of each polynomial is r. q.e.d.

Proposition 45 (Differences between sums of n addends) *The result is a vector whose components contain monomials of degree $r-1$ and coefficients 1.*

$$\vec{S}(1,1,n) - \vec{S}(0,1,n) = \vec{V}(n) - \vec{V}(0)$$

It applies: D:8,10; is applied in: P:46.

This is obtained by telescopic simplification from P:39. The result is therefore a vector with components: $0, n, n^2..., n^{m-1}$, i.e. a vector of monomials of degree r-1 and coefficients 1 q.e.d.

Proposition 46 (Differenza tra le matrici gemelle)

$$[G_1 - G_0]_{r,c} = \begin{cases} 1 & \textit{if } r = c-1 \\ 0 & \textit{otherwise} \end{cases}$$

It applies: D:7,10; P:44,45; is applied in P:47,48.

For D:10 we have $\vec{S}(0,1,n) = G_0 n\vec{V}(n) =$ e $\vec{S}(1,1,n) = G_1 n\vec{V}(n)$ passing to the r-th component:

$$S_{r-1}(0,1,n) = [G_0]_{r,1}n + [G_0]_{r,2}n^2 + ... + [G_0]_{r,r}n^r$$
$$S_{r-1}(1,1,n) = [G_1]_{r,1}n + [G_1]_{r,2}n^2 + ... + [G_1]_{r,r}n^r$$

As demonstrated (P:45) the difference between the two polynomial vectors is the vector $0, n, n^2..., n^{m-1}$. So for $r > 1$ it results:

$$S_{r-1}(1,1,n) - S_{r-1}(0,1,n) = n^{r-1}$$

therefore the differences cancel all the monomials of the two polynomials except one having degree $r-1$, i.e. the second having degree (P:44). This proves that the two vectors considered with their polynomials, after the first identical component, differ only in the second monomials in degree. These are placed, in G_1 and in G_0, in the diagonal immediately below the main one which instead contains the prime coefficients. q.e.d.

7.1 Second Bernoulli numbers

Proposition 47 (Differences between the $\vec{B}$ and $\vec{B}^+$ variants of Bernoulli numbers) *First and second Bernoulli numbers.* [2]

$$B^+_{r-1} = B_{r-1} \textit{ if } r \neq 2 \textit{ otherwise } B^+_1 = -B_1 = \frac{1}{2}$$

It applies: D:7,9; P:31,41,46; is applied in: P:55,56; S:10.

[2]OEIS integer sequences A027641/A027642 (first variant) and A164555/A027642 (second variant)

For P:46 the only difference between the first column of G_1 and the corresponding one in G_0 is in the second element which also belongs to the diagonal differentiating the two matrices.
For P:46 it is $[G_1]_{2,1} - [G_0]_{2,1} = 1$ while for P:41 it results in $[G_1]_{2,1} = -[G_0]_{2,1}$. From this we get $[G_0]_{2,1} = -\frac{1}{2}$ and $[G_1]_{2,1} = \frac{1}{2}$.
For P:31 and for D:9 we have $[G_0]_{r,1} = B_{r-1}(0) = B_{r-1}$ while $[G_1]_{r,1} = B_{r-1}(1) = B^+_{r-1}$ So the first column of the matrix G_1 contains the Bernoulli numbers $\vec{B}(0) = \vec{B}$ with the second component $B_1 = -\frac{1}{2}$ while the first column of G_1 contains the variant $\vec{B}(1) = \vec{B^+}$ with $B_1^+ = \frac{1}{2}$ q.e.d.

7.2 Zero Bernoulli numbers

Proposition 48 (Zeros) *in twin matrices (of order m)*

$$\text{if} \quad [G_0]_{r,c} = [G_1]_{r,c} = x \quad \text{and} \quad r + c \text{ is odd then } x = 0$$

It applies: D:7,11; P:41,46; is applied in: P:49.

In fact, being G_0 and G_1 in an alternation relation (P:41) all the elements such that the number of the row plus that of the column is odd must be opposite. On the other hand it is known that, apart from the diagonal of the second degree coefficients, the elements of the two matrices coincide (P:46) so that there remains only the possibility that they are null, q.e.d.

Proposition 49 (Zero bernoulli number)

$$B_j = 0 \quad \text{for odd } j > 1$$

It applies: D:7,9; P:31,48; is applied in: P:55.

For P:31 we have $B_{r-1} = [G_0]_{r,1}$ the sum of the indices $r + 1$, when the condition is satisfied, is odd and for P :48 the elements corresponding to these indices are null. q.e.d.

7.3 Bernoulli numbers from Pascal's triangle

7.4 H, a binomial Hessemberg matrix and its variants

Definition 17 (Hessemberg matrix) *From Pascal's triangle*

$$[H]_{r,c} = \binom{1+r}{c-1} \quad \text{if } c \leq 1 + r \quad \text{otherwise} \quad 0$$

is applied in: E:28; P:50,51,52.

Proposition 50 (For the Bernoulli numbers)

$$B_m = (-1)^m \frac{|H|}{(m+1)!} \qquad m \in \mathbb{N}^+$$

It applies: D:9,17; P:31; is applied in: E:29; P:31,51.

This formula (discovered by the author in 2007 [8]), directly relates the Bernoulli numbers to Pascal's triangle. We have seen that the matrix A which this time is assumed to have order $m+1$ has an inverse matrix A^{-1} which has in the first column the first degree coefficients of the polynomials coinciding with the Bernoulli numbers

P:31. Bearing in mind the method of algebraic complements for calculating the inverse matrix, we have:

$$[A^{-1}]_{m+1,1} = B_m = (-1)^{m+2}\frac{|A_{1,m+1}|}{|A|} = (-1)^m\frac{|H|}{(m+1)!}$$

where is it:

$[A^{-1}]_{m+1,1}$ indicates the corresponding element of $[A]_{m+1,1}$ in the inverse matrix

$|A| = (m+1)!$ is the determinant of the triangular matrix of order m+1

$|A_{1,m+1}| = |H|$ is the algebraic complement of order m (obtained by deleting the first row and the last column) relative to the element $[A]_{1,m+1}$ corresponding to $[A]_{m+1,1}$ in the transposition for calculating the inverse.

So the formula gives the Bernoulli numbers q.e.d.

Note This formula gives the most commonly used Bernoulli numbers with $B_1 = -\frac{1}{2}$. By omitting the factor $(-1)^n$ the sign of all numbers with odd index is changed, i.e. only B_1 given that, as demonstrated, the others are all null. It is therefore convenient to simplify the formula by considering the variant with $B_1 = \frac{1}{2}$

Example 28 (H matrix) *order m=10 components*

$$H = \begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 0 & 0 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 0 & 0 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 0 & 0 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 0 \\ 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 \\ 1 & 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 \end{vmatrix} = 3024000$$

It applies: D:17; It is applied in: E:29.

Example 29 (Bernoulli numbers) *from Pascal's triangle*

$$B_{10} = \frac{H}{11!} = \frac{3024000}{39916800} = \frac{5}{66}$$

It applies: E:28; P:50.

Proposition 51 (Only matrix) *Variant* $X = (N+U)^{-1}H$

$$B_n = |X| \ \textit{for}\ n > 0 \quad \textit{with}\ [X]_{r,c} = \begin{cases} 0 & \textit{if}\ c > 1+r \\ \binom{r+1}{c-1}\frac{1}{r+1} & \textit{if}\ c \le 1+r \end{cases}$$

It applies: D:1,2,9,17; P:50; is applied in: P:52.

By the properties of determinants, dividing the rows by $r+1$ *with* $r = 1...n$ divides the determinant of H by $(n+1)!$ q.e.d.

Proposition 52 (Numerator and denominator) *Variant with matrix C of order n to obtain numerator and denominator, reduced to lowest terms, of Bernoulli numbers from Pascal's triangle*

$$B_n = \frac{|C|}{\prod_{k=1}^{n}[C]_{k,2}} \quad \textit{with} \quad [C]_{r,c} = \begin{cases} 0 & \textit{if}\ c > 1+r \\ \binom{r+1}{c-1} & \textit{if}\ r+1\ \textit{is prime and}\ r\ \textit{divides}\ n \\ \binom{r+1}{c-1}\frac{1}{r+1} & \textit{otherwise} \end{cases}$$

It applies: D:9,17; P:51; is applied in: E:30.

It is obtained from the previous variant by conditioning the multiplication of $\frac{1}{r+1}$ to the condition of the Von Staudt-Clausen theorem. By the same theorem the obtained numerator and denominator are reduced to their minima

Example 30 (Numerator and denominator) *reduced to lowest terms in Bernoulli numbers extracted from Pascal's triangle*

$$num(B_{10}) = \begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & \frac{3}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & 1 & \frac{5}{2} & \frac{10}{3} & \frac{5}{2} & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{7} & 1 & 3 & 5 & 5 & 3 & 1 & 0 & 0 & 0 \\ \frac{1}{8} & 1 & \frac{7}{2} & 7 & \frac{35}{4} & 7 & \frac{7}{2} & 1 & 0 & 0 \\ \frac{1}{9} & 1 & 4 & \frac{28}{3} & 14 & 14 & \frac{28}{3} & 4 & 1 & 0 \\ \frac{1}{10} & 1 & \frac{9}{2} & 12 & 21 & \frac{126}{5} & 21 & 12 & \frac{9}{2} & 1 \\ 1 & 11 & 55 & 165 & 330 & 462 & 462 & 330 & 165 & 55 \end{vmatrix} = 5$$

$den(B_{10}) = 2 \cdot 3 \cdot 11 = 66$

It applies: P:52.

8 So-called Faulhaber's formula

Proposition 53 (Numerator and denominator) *reduced to lowest terms in Bernoulli numbers extracted from Pascal's triangle*

$$\vec{S}(h,d,n) = G(h,d)\ n\vec{V}(n) \qquad G(h,d) = \tilde{V}(d) \circ N^{-1} \circ Z \circ \widehat{B}(\frac{h}{d})$$

m-order matrices, product rows by column:

$$S_{m-1}(h,d,n) = \sum_{k=0}^{n-1}(h+dk)^{m-1} = \frac{d^{m-1}}{m}\sum_{k=1}^{m}\binom{m}{k}B_{m-k}\left(\frac{h}{d}\right)n^k$$

It applies: D:1,5,7,8,9,10,12,13,14; P:30,39; is applied in: P:55,56,58.

The modification of the statement of the G theorem (P:39) is based on P:30 The following formula is obtained by passing to the m-th component of the vector of sums of powers q.e.d.

Proposition 54 (G theorem third formulation) *Complete binomial coefficients*

$$\vec{S}(h,d,n) = G(h,d)\ n\vec{V}(n) \qquad G(h,d) = \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}$$

m-order matrices, product rows by column:

$$\sum_{k=0}^{n-1}(h+dk)^{m-1} = d^{m-1}\sum_{k=1}^{m}\frac{1}{k}\binom{m-1}{k-1}B_{m-k}\left(\frac{h}{d}\right)n^k$$

It applies: D:1,3,7,8,9,10,12,13,14; P:29,39,57,61.

The modification of the statement of theorem G (P:39) is based on P:29 The following formula is obtained by passing to the m-th component of the vector of sums of powers q.e.d.

Proposition 55 (So-called Faulhaber formulas) *Particular case of the second formulation of the theorem G. Generally with "Faulhaber formula" we indicate the polynomial formulas resolving sums of n powers of successive integers starting from 1.*

$$\vec{S}(1,1,n) = N^{-1}(Z \circ \widehat{B}^{+})n\vec{V}(n)$$

m-th component:

$$S_{m-1}(1,1,n) = \sum_{k=0}^{n-1}(1+k)^{m-1} = \sum_{k=1}^{n} k^{m-1} = \frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}B^{+}_{m-k}n^{k}$$

Sometimes it is preferred to avoid the use of the variant $\vec{B}^{+}$ of the Bernoulli numbers (P:47 different from this one only for the second element $B_1 = \frac{1}{2}$. However, this forces us to artificially complicate the formula by changing the sign of all Bernoulli numbers of odd index, given that all the others are zero(P:49)

$$S_{m-1}(1,1,n) = \frac{1}{m}\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}B_{m-k}n^{k}$$

It applies: D:1,5,7,8,9,10,12,13,14; P:47,49,53.

Proposition 56 (So-called Faulhaber formulas) *Particular case of the second formulation of the theorem G. Sometimes it is preferable to give formulas resolving sums of n powers of successive integers starting from 0. These too are called Faulhaber formulas.*

$$\vec{S}(0,1,n) = N^{-1}(Z \circ \widehat{B})n\vec{V}(n)$$

m-th component:

$$S_{m-1}(0,1,n) = \sum_{k=0}^{n-1}(1+k)^{m-1} = \sum_{k=1}^{n} k^{m-1} = \frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}B^{+}_{m-k}n^{k}$$

It applies: D:1,5,7,8,9,10,12,13,14; P:29,47,53;
is applied in: E:31.

Example 31 ***m=4*** $\vec{S}(0,1,n) = N^{-1}(Z \circ \widehat{B})n\vec{V}(n) =$

$$\begin{bmatrix} \frac{1}{1} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \binom{1}{1}B_0 & 0 & 0 & 0 \\ \binom{2}{1}B_1 & \binom{2}{2}B_0 & 0 & 0 \\ \binom{3}{1}B_2 & \binom{3}{2}B_1 & \binom{3}{3}B_0 & 0 \\ \binom{4}{1}B_3 & \binom{4}{2}B_2 & \binom{4}{3}B_1 & \binom{4}{4}B_0 \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{1}\binom{1}{1}B_0 & 0 & 0 & 0 \\ \frac{1}{2}\binom{2}{1}B_1 & \frac{1}{2}\binom{2}{2}B_0 & 0 & 0 \\ \frac{1}{3}\binom{3}{1}B_2 & \frac{1}{3}\binom{3}{2}B_1 & \frac{1}{3}\binom{3}{3}B_0 & 0 \\ \frac{1}{4}\binom{4}{1}B_3 & \frac{1}{4}\binom{4}{2}B_2 & \frac{1}{4}\binom{4}{3}B_1 & \frac{1}{4}\binom{4}{4}B_0 \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{1}\binom{1}{1}B_0 n \\ \frac{1}{2}\binom{2}{1}B_1 n + \frac{1}{2}\binom{2}{2}B_0 n^2 \\ \frac{1}{3}\binom{3}{1}B_2 n + \frac{1}{3}\binom{3}{2}B_1 n^2 + \frac{1}{3}\binom{3}{3}B_0 n^3 \\ \frac{1}{4}\binom{4}{1}B_3 n + \frac{1}{4}\binom{4}{2}B_2 n^2 + \frac{1}{4}\binom{4}{3}B_1 n^3 + \frac{1}{4}\binom{4}{4}B_0 n^4 \end{bmatrix} =$$

$$= \begin{bmatrix} S_0(0,1,n) \\ S_1(0,1,n) \\ S_2(0,1,n) \\ S_3(0,1,n) \end{bmatrix} = \begin{bmatrix} \frac{1}{1}\sum_{k=1}^{1}\binom{1}{k}B_{1-k}n^k \\ \frac{1}{2}\sum_{k=1}^{2}\binom{2}{k}B_{2-k}n^k \\ \frac{1}{3}\sum_{k=1}^{3}\binom{3}{k}B_{3-k}n^k \\ \frac{1}{4}\sum_{k=1}^{4}\binom{4}{k}B_{4-k}n^k \end{bmatrix} = \begin{bmatrix} n \\ -\frac{1}{2}n + \frac{1}{2}n^2 \\ \frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3 \\ \frac{1}{4}n^2 - \frac{1}{2}n^3 + \frac{1}{4}n^4 \end{bmatrix}$$

It applies: D:4,6; P:39.

Proposition 57 (So-called Faulhaber's formulas) *Special case of the third formulation of G theorem.*

$$\vec{S}(0,1,n) = (T \circ \widehat{B})N^{-1}n\vec{V}(n)$$

m-th component:

$$S_{m-1}(0,1,n) = \sum_{k=0}^{n-1}(1+k)^{m-1} = \sum_{k=1}^{n} k^{m-1} = \sum_{k=1}^{m} \frac{1}{k}\binom{m-1}{k-1} B_{m-k} n^k$$

It applies: D:1,3,7,8,9,10,12,13,14; P: 54.

Example 32 ***m=4*** $\quad \vec{S}(0,1,n) = (T \circ \vec{B})N^{-1}n\vec{V}(n) =$

$$\begin{bmatrix} \binom{0}{0}B_0 & 0 & 0 & 0 \\ \binom{1}{0}B_1 & \binom{1}{1}B_0 & 0 & 0 \\ \binom{2}{0}B_2 & \binom{2}{1}B_1 & \binom{2}{2}B_0 & 0 \\ \binom{3}{0}B_3 & \binom{3}{1}B_2 & \binom{3}{2}B_1 & \binom{3}{3}B_0 \end{bmatrix} \begin{bmatrix} \frac{1}{1} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{1}\binom{0}{0}B_0 & 0 & 0 & 0 \\ \frac{1}{1}\binom{1}{0}B_1 & \frac{1}{2}\binom{1}{1}B_0 & 0 & 0 \\ \frac{1}{1}\binom{2}{0}B_2 & \frac{1}{2}\binom{2}{1}B_1 & \frac{1}{3}\binom{2}{2}B_0 & 0 \\ \frac{1}{1}\binom{3}{0}B_3 & \frac{1}{2}\binom{3}{1}B_2 & \frac{1}{3}\binom{3}{2}B_1 & \frac{1}{4}\binom{3}{3}B_0 \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1}{1}\binom{0}{0}B_0 n \\ \frac{1}{1}\binom{1}{0}B_1 n + \frac{1}{2}\binom{1}{1}B_0 n^2 \\ \frac{1}{1}\binom{2}{0}B_2 n + \frac{1}{2}\binom{2}{1}B_1 n^2 + \frac{1}{3}\binom{2}{2}B_0 n^3 \\ \frac{1}{1}\binom{3}{0}B_3 n + \frac{1}{2}\binom{3}{1}B_2 n^2 + \frac{1}{3}\binom{3}{2}B_1 n^3 + \frac{1}{4}\binom{3}{3}B_0 n^4 \end{bmatrix} =$$

$$= \begin{bmatrix} S_0(0,1,n) \\ S_1(0,1,n) \\ S_2(0,1,n) \\ S_3(0,1,n) \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{1} \frac{1}{k}\binom{0}{k-1}B_{1-k}n^k \\ \sum_{k=1}^{2} \frac{1}{k}\binom{1}{k-1}B_{2-k}n^k \\ \sum_{k=1}^{3} \frac{1}{k}\binom{2}{k-1}B_{3-k}n^k \\ \sum_{k=1}^{4} \frac{1}{k}\binom{3}{k-1}B_{4-k}n^k \end{bmatrix} = \begin{bmatrix} n \\ -\frac{1}{2}n + \frac{1}{2}n^2 \\ \frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3 \\ \frac{1}{4}n^2 - \frac{1}{2}n^3 + \frac{1}{4}n^4 \end{bmatrix}$$

It applies: D:6,9; P:56.

9 Various properties of Bernoulli polynomials

9.1 Sums of powers with bases in arithmetic progression

Proposition 58 (Translative form)

$$\sum_{k=0}^{n-1}(h+dk)^{m-1} = \frac{d^{m-1}}{m}\Big(B_m(\frac{h}{d}+n) - B_m(\frac{h}{d})\Big)$$

It applies: D:9; P:16,53.

Finally, it is possible to intervene on the generalized Faulhaber formula (P:53) by applying the translation property (P:16) to obtain a simpler form, but which does not explicitly provide the coefficients of the polynomials in the canonical form .

$$\sum_{k=1}^{m}\binom{m}{k}B_{m-k}(\frac{h}{d})n^k = -B_m(\frac{h}{d}) + \sum_{k=0}^{m}\binom{m}{k}B_{m-k}(\frac{h}{d})n^k = B_m(\frac{h}{d}+n) - B_m(\frac{h}{d})$$

q.e.d.

9.2 Symmetric property

Proposition 59 (Symmetries in Bernoulli polynomials) *Relationship between* $\vec{B}(\frac{1}{2}+y)$ *and* $\vec{B}(\frac{1}{2}-y)$*. We will prove that*

$$\vec{B}(x) = J\vec{B}(1-x)$$

and then also:

$$B_j(x) = (-1)^j B_j(1-x)$$

It applies: D:2,8,9,11; P:31,42.

For P:41 there are semi-oppositions $G(x,1)$ and $G(1-x,1)$ and it results $G(x,1) = JG(1-x,1)J$ Multiplying the two members on the right by the vector $\vec{V}(0)$ to extract the first column we have $G(x,1)\vec{V}(0) = JG(1-x,1)J\vec{V}(0)$. Since $J\vec{V}(0) = \vec{V}(0)$ we have:

$$G(x,1)\vec{V}(0) = JG(1-x,1)\vec{V}(0)$$

For P:31 the first column corresponds to the Bernoulli polynomials for which the vector form of the thesis is proved. Moving on to the components $B_{r-1}(x) = (-1)^{r+1}B_{r-1}(1-x)$ Having established that $r+1$ and $r-1$ are both even or both odd, therefore $(-1)^{r+1} = (-1)^{r-1}$ and set $j = r-1$ you get the thesis. q.e.d.

9.3 Derivatives of the vector S(h,d,x)

Proposition 60 (Derivatives of the vector S(h,d,x) with respect to x)

$$\frac{\partial \vec{S}(h,d,x)}{\partial x} = \vec{B}(h+xd)$$

Special cases:
Per $d = 1$ *:* $\frac{\partial \vec{S}(h,1,x)}{\partial x} = \vec{B}(x+h)$
Per $h = 0$ *e* $d = 1$ *:* $\frac{\partial \vec{S}(0,1,x)}{\partial x} = \vec{B}(x)$
It applies: D:1,2,3,8,9,12,13,14,15; E:2; P:20,29.

Per P:29

$$G(h,d) = \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}$$

therefore for D:7

$$\vec{S}(h,d,x) = \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}x\vec{V}(x)$$

So by differentiating we have:

$$\frac{\partial \vec{S}(h,d,x)}{\partial x} = \frac{\partial \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}x\vec{V}(x)}{\partial x} = \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}\frac{\partial (x\vec{V}(x))}{\partial x} =$$

As also shown in the example E:2, the derivative of $(x, x^2, x^3, ...)$ is $(1, 2x, 3x^2, 3x^3...)$ i.e. $N\vec{V}(x)$ then also applying P:20

$$= \tilde{V}(d)(T \circ \widehat{B}(\frac{h}{d}))N^{-1}N\vec{V}(x) = \tilde{V}(d)\vec{B}(\frac{h}{d}+x) = \vec{B}(h+xd)$$

Having considered $N^{-1}N = U$ and having finally multiplied the diagonal matrix by the vector B. q.e.d.

Proposition 61 (Deriving the vector S with respect to h)

$$B_{r-1}(x) = \frac{B'_r(x)}{r}$$

It applies: D:1,3,8,9,12,13,14,15; P:29,39,54.

For P:29

$$G(h,1) = (T \circ \widehat{B}(h))N^{-1}$$

therefore for P:39 and for D:7

$$\vec{S}(h,1,x) = \sum_{k=0}^{n-1} \vec{V}(h+k) = (T \circ \widehat{B}(h))N^{-1}x\vec{V}(x) =$$

Considering the m-th component as in P:54 but with $d = 1$ we have $S_{m-1}(h,1,n) =$

$$\sum_{k=0}^{n-1}(h+k)^{m-1} = \sum_{k=1}^{m} \frac{1}{k}\binom{m-1}{k-1} B_{m-k}(h)n^k$$

deriving the two members of the identity with respect to h:

$$(m-1)\sum_{k=0}^{n-1}(h+k)^{m-2} = \sum_{k=1}^{m} \frac{1}{k}\binom{m-1}{k-1} B'_{m-k}(h)n^k$$

Taking into account the fact that for $k = m$ the coefficient of the higher degree monomial is $B'_0(h) = 0$ given that $B_0(h) = 1$ we have:

$$(m-1)\sum_{k=0}^{n-1}(h+k)^{m-2} = \sum_{k=1}^{m-1} \frac{1}{k}\binom{m-1}{k-1} B'_{m-k}(h)n^k$$

Assuming $m > 1$, multiply the two sides by the reciprocal of $m-1$ and taking into account that it results $\frac{1}{(m-1)k}\binom{m-1}{k-1} = \frac{1}{(m-1)k}\frac{(m-1)!}{(k-1)!(m-k)!} = \frac{1}{k}\frac{(m-2)!}{(k-1)!(m-k-1)!}\frac{1}{m-k} = \frac{1}{k}\binom{m-2}{k-1}\frac{1}{m-k}$ we have:

$$\sum_{k=0}^{n-1}(h+k)^{m-2} = \sum_{k=1}^{m-1} \frac{1}{k}\binom{m-2}{k-1} \frac{B'_{m-k}}{m-k}(h)n^k$$

on the other hand for P:39 and for D:7 it must also be:

$$\sum_{k=0}^{n-1}(h+k)^{m-2} = \sum_{k=1}^{m-1} \frac{1}{k}\binom{m-2}{k-1} B_{m-k-1}(h)n^k$$

therefore the coefficients of the two polynomials must coincide and, setting $r = m - k$ it must therefore be $B_{r-1} = \frac{B'_r}{r}$ with $r = 1...m-1$ q.e.d.

9.4 Generator function theorem

Proposition 62 (Generator function)

$$\frac{xe^{hx}}{e^x - 1} = \sum_{k=0}^{\infty} B_k(h)\frac{x^k}{k!}$$

Special cases h=0 e h=1:

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \qquad\qquad \frac{xe^x}{e^x-1} = \sum_{k=0}^{\infty} B_k^+ \frac{x^k}{k!}$$

It applies: D:9; P:15.

Expansion in infinite series:

$$e^{hx} = \sum_{k=0}^{\infty} \frac{h^k x^k}{k!} = 1 + hx + h^2 \frac{x^2}{2!} + h^3 \frac{x^3}{3!} + h^4 \frac{x^4}{4!} + ...$$

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

substituting we have:

$$\frac{x(1 + hx + h^2 \frac{x^2}{2!} + h^3 \frac{x^3}{3!} + h^4 \frac{x^4}{4!} + ...)}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...} = \sum_{k=0}^{\infty} B_k(h) \frac{x^k}{k!}$$

collecting and simplifying

$$\frac{1 + \frac{h}{p}x + \frac{h^2}{p^2}\frac{x^2}{2!} + \frac{h^3}{p^3}\frac{x^3}{3!} + \frac{h^4}{p^4}\frac{x^4}{4!} + ...}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + ...} = \sum_{k=0}^{\infty} B_k(h) \frac{x^k}{k!}$$

Multiplying the two sides of this equation by the denominator of the first expression we obtain the first side

$$1 + \frac{h}{p}x + \frac{h^2}{p^2}\frac{x^2}{2!} + \frac{h^3}{p^3}\frac{x^3}{3!} + \frac{h^4}{p^4}\frac{x^4}{4!} + ...$$

and to the second member:

$$(1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + ...) \sum_{k=0}^{\infty} B_k(h) \frac{x^k}{k!}$$

that gives:

$$\begin{array}{l}
\frac{B_0(h)}{1!0!}x^0 + \frac{B_1(h)}{1!1!}x^1 + \frac{B_2(h)}{1!2!}x^2 + \frac{B_3(h)}{1!3!}x^3 + \frac{B_4(h)}{1!4!}x^4 + ... \\
\frac{B_0(h)}{2!0!}x^1 + \frac{B_1(h)}{2!1!}x^2 + \frac{B_2(h)}{2!2!}x^3 + \frac{B_3(h)}{2!3!}x^4 + \frac{B_4(h)}{2!4!}x^5 + ... \\
\frac{B_0(h)}{3!0!}x^2 + \frac{B_1(h)}{3!1!}x^3 + \frac{B_2(h)}{3!2!}x^4 + \frac{B_3(h)}{3!3!}x^5 + \frac{B_4(h)}{3!4!}x^6 + ... \\
\frac{B_0(h)}{4!0!}x^3 + \frac{B_1(h)}{4!1!}x^4 + \frac{B_2(h)}{4!2!}x^5 + \frac{B_3(h)}{4!3!}x^6 + \frac{B_4(h)}{4!4!}x^7 + ... \\
\frac{B_0(h)}{5!0!}x^4 + \frac{B_1(h)}{5!1!}x^5 + \frac{B_2(h)}{5!2!}x^6 + \frac{B_3(h)}{5!3!}x^7 + \frac{B_4(h)}{5!4!}x^8 + ...
\end{array}$$

Arranging the monomials in increasing order of degree along the diagonals we obtain the infinite polynomial:

$$B_0(h) + \frac{1}{(1+1)!}\sum_{j=0}^{1} B_j(h)\binom{1+1}{j}x^1 + \frac{1}{(2+1)!}\sum_{j=0}^{2} B_j(h)\binom{2+1}{j}x^2 +$$

$$+\frac{1}{(3+1)!}\sum_{j=0}^{3} B_j(h)\binom{3+1}{j}x^3 + \frac{1}{(4+1)!}\sum_{j=0}^{4} B_j(h)\binom{4+1}{j}x^4 + ... =$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\sum_{j=0}^{k} B_j(h)\binom{k+1}{j}x^k = \sum_{k=0}^{\infty} \frac{1}{(k+1)}\sum_{j=0}^{k} B_j(h)\binom{k+1}{j}\frac{x^k}{k!}$$

Comparing this second member with the first previously mentioned we have:

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)}\sum_{j=0}^{k} B_j\left(h\right)\binom{k+1}{j}\frac{x^k}{k!} = \sum_{k=0}^{\infty} h^k \frac{x^k}{k!}$$

from which the identity of the coefficients of $\frac{x^k}{k!}$:

$$= \frac{1}{(k+1)} \sum_{j=0}^{k} B_j(h) \binom{k+1}{j} = h^k$$

multiplying the two sides by $m = k+1$ and changing the order of the factors:

$$\sum_{j=0}^{m-1} \binom{m}{j} B_j(h) = mh^{m-1}$$

Finally expressing the same sum through the index j shifted by one unit

$$\sum_{j=1}^{m} \binom{m}{j-1} B_{j-1}(h) = mh^{m-1}$$

we can easily recognize the definition given for Bernoulli polynomials (P:15). q.e.d.

10 A little of history

Faulhaber's formula solves the problem of the sum of powers of successive integers [4] in a direct and general way. A classic problem that has captured the interest of mathematicians for millennia. Pascal, for example, in 1634 had identified an identity for the recursive resolution of the problem (P:35). In P:55 we proved Faulhaber's formula also in the form that today seems to be more commonly adopted.

However, the choice of expressing it through the (first) Bernoulli numbers appears artificial due to that negative factor raised to the power which multiplies many times uselessly for the sole purpose of changing the sign of B_1. More natural, as shown, to choose the second Bernoulli numbers, the almost identical variant but with the opposite B_1 (P:47).

Indeed, the numbers considered by Bernoulli were neither the first nor the second sequence as they began, and continued to start for over a century, from B_2 immediately after the controversial B_1. This numerical sequence was published by the Bernoulli family in *Ars Conjectandi* [2] in 1713 eight years after the death of Jacob, author of the work. In the chapter *Summae potestatum* also appeared the formula for constructing the polynomials calculating the sums of powers of successive integers as a function of that particular unpublished and mysterious numerical sequence. Here, albeit with modern symbols, is that formula that is only apparently different from the most common. (P:55):

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1} n^{m+1} + \frac{1}{2} n^m + \sum_{k=2}^{n} \frac{m^{\underline{k-1}}}{k!} B_k n^{m-k+1}$$

The underlined exponent indicates the number of factors in the *decreasing factorial*. A few years later De Moivre and Euler began calling these numbers by their current name. Whereas the formula was named after Faulhaber, who in his life had shown an extraordinary interest and exceptional virtuosity in solving particular cases of this problem up to the seventeenth degree of the exponents. Even Bernoulli had given Faulhaber part of the credit for the formula found but not yet proven. After over a century it was Carl Jacobi (1804-1851) [3] who succeeded. The proof was based on developments in mathematical analysis which had revealed that:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{1}$$

imilarly, the Bernoulli numbers were also defined analytically. In 1842, in an age prior to that of computers, Ada Lovelace devised a program, the first in history, which was supposed to calculate the Bernoulli numbers with the future analytical engine designed by Charles Baggage. For this purpose you essentially used the same recursive algorithm seen in P: 16 [10] This, however, was explained by resorting to the generating function (note G [5]). The so-called Bernoulli polynomials were taken into consideration by Leonardo Euler (1707-1783) even if the first to call them by this name was J.L. Raabe in 1851 [16] Also these polynomials are traditionally defined through the related generating function.

In 1982 A.W.F. Edwards published an article [6] in which, starting from an identity used by Pascal in 1654 [7] to recursively solve the problem of the sum of the powers of successive integers, he shows that the coefficients of the polynomials solving the problem can be found by inverting a matrix easily obtainable from Pascal's triangle
Around 1990 the author accidentally discovered the same relationship while preparing a matrix exercise for his [8] high school students. About fifteen years later, retired and still unable to find the topic in any publication, he tried to prove what he discovered. He succeeded and published demonstrations and history of the discovery on the web [8]. In 2017, after about a decade without significant developments, the author renewed his interest in the problem by deciding to take an alternative path to the analytic one historically traced by Carl Jacobi. So he began to prove formulas for powers of successive integers and properties of Bernoulli numbers using matrices instead of the usual mathematical analysis. The same year he became aware of the works of Edwards [7] and published some results of the path undertaken on the web in English [9] capturing the interest of researchers [15]. The author organizes the numerous results of his research in the form of a treatise which he will rewrite several times for new discoveries and for the continuous search for more suitable symbols and better organization. In 2021 a small part of the work done is published in Archimede [13] and the following year in MatematicaMente. [14].
Some fruits of the research undertaken are proposed in this article.

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