Matrices for a quick proof of the classical problem of sums of powers of successive integers

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1 A new solution to an old problem

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \cdots + n^p$$

We will consider a positive integer number m which will be implied and will not appear in the symbols adopted but which can be chosen as desired. It will be $m = p + 1$ because the exponents considered will go from 0 to p. For example for m=6 we have:

$$\vec{S}(n) = \begin{bmatrix} \sum_{k=1}^{n} k^0 \\ \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \\ \sum_{k=1}^{n} k^4 \\ \sum_{k=1}^{n} k^5 \end{bmatrix} \quad \vec{V}(j) = \begin{bmatrix} 1 \\ j \\ j^2 \\ j^3 \\ j^4 \\ j^5 \end{bmatrix} \quad j\vec{V}(j) = \begin{bmatrix} j \\ j^2 \\ j^3 \\ j^4 \\ j^5 \\ j^6 \end{bmatrix} \quad \vec{V}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Naturally for any m chosen it will result: $\sum_{k=1}^{n} \vec{V}(k) = \vec{S}(n)$ Knowledge of the development of Newton's binomial and the row-column product is sufficient to understand the following easily generalizable identity:

$$\begin{bmatrix} j-(j-1) \\ j^2-(j-1)^2 \\ j^3-(j-1)^3 \\ j^4-(j-1)^4 \\ j^5-(j-1)^5 \\ j^6-((j-1)^6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ j \\ j^2 \\ j^3 \\ j^4 \\ j^5 \end{bmatrix}$$

By decomposing the first member vector of this identity and indicating the matrix with alternating signs obtainable from the Pascal triangle with $\overline{A}$ we can generalize this identity by writing:

$$j\vec{V}(j) - (j-1)\vec{V}(j-1) = \overline{A}\vec{V}(j)$$

adding member to member for i from 1 to n and simplifying (telescopically) the opposites we get:

$$\sum_{k=1}^{n}(j\vec{V}(j) - (j-1)\vec{V}(j-1)) = \sum_{k=1}^{n}\overline{A}\vec{V}(j)$$

$$n\vec{V}(n) - 0\vec{V}(0) = \overline{A}\sum_{k=1}^{n}\vec{V}(j)$$

$$n\vec{V}(n) = \overline{A}\vec{S}(n)$$

from which multiplying the two sides of the equation on the left by the inverse matrix we obtain:

$$\vec{S}(n) = \overline{A}^{-1}n\vec{V}(n)$$

Example 1 *Choosing m=11 and inverting the matrix* $\overline{A}$ *we obtain the famous polynomials published in 1713 in Ars Conjectandi:*

$$\begin{bmatrix} \sum_{k=1}^{n} k^0 \\ \sum_{k=1}^{n} k^1 \\ \sum_{k=1}^{n} k^2 \\ \sum_{k=1}^{n} k^3 \\ \sum_{k=1}^{n} k^4 \\ \sum_{k=1}^{n} k^5 \\ \sum_{k=1}^{n} k^6 \\ \sum_{k=1}^{n} k^7 \\ \sum_{k=1}^{n} k^8 \\ \sum_{k=1}^{n} k^9 \\ \sum_{k=1}^{n} k^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 & \\ 0 & \frac{1}{12} & 0 & -\frac{7}{24} & 0 & \frac{1}{12} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 & \\ -\frac{1}{30} & 0 & \frac{2}{9} & 0 & -\frac{7}{15} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{9} & 0 & 0 & \\ 0 & -\frac{3}{20} & 0 & \frac{1}{2} & 0 & -\frac{7}{10} & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{10} & 0 & \\ \frac{5}{66} & 0 & -\frac{1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & \frac{1}{2} & \frac{1}{11} & \end{bmatrix} \begin{bmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ n^9 \\ n^{10} \\ n^{11} \end{bmatrix}$$

2 Bibliografia

References

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