Radicands to complete Faulhaber polynomials

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Abstract

Jhoann Faulhaber, a 17th century German mathematician, in his studies found polynomial functions that, in the case of odd exponents, compute the sums of powers of successive integers in the form, $S_1(p), S_3(p), S_5(p), \ldots$ with $p = S_1(p)$ It turned out to be impossible to do the same thing with even indices. This becomes possible, however, if one considers more general functions. In this article, by connecting two articles by A.W.F.Edwards and N.Derby, it is shown how to obtain also $S_2(p), S_4(p), S_6(p), \ldots$

1 Definitions

Let it be:
-variables

$$S_i = \sum_{k=1}^{n} k^i \qquad S_1 = \frac{n(n+1)}{2} = p \qquad u = 2p$$

-matrices and vectors

$$[A]_{r,c} = \binom{r}{c-1} \quad \text{if } r \geq c \text{, otherwise } 0$$

$$K = A^{-1}$$

$$[\tilde{X}]_{r,c} = X_{r+1,2c-r+1} \quad \text{if } \ \frac{1}{2}r \leq c \leq r, \ \text{ otherwise } 0$$

$$[\vec{S}]_r = S_r^2 \qquad [\vec{D}]_r = S_{2r+1}$$

$$[\vec{U}]_r = u^{i+1} \qquad [\vec{P}]_r = p^{i+1}$$

$$[Q]_{r,c} = 2^{r+1} \quad \text{if } \ r = c, \ \text{ otherwise } 0$$

1.1 Exemples

In these examples the number m of the components of the vectors and order of the square matrices, which can be any positive integer, is chosen equal to six (m=6)

$$\vec{S} = \begin{bmatrix} S_1^2 \\ S_2^2 \\ S_3^2 \\ S_4^2 \\ S_5^2 \\ S_6^2 \end{bmatrix} \qquad \vec{D} = \begin{bmatrix} S_3 \\ S_5 \\ S_7 \\ S_9 \\ S^{11} \\ S^{13} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \end{bmatrix} \qquad A^{-1} = K = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}$$

Considering that the last row of the two matrices in the following case (m=7) is respectively:

$$\begin{bmatrix} 1 & 7 & 21 & 35 & 35 & 21 & 7 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{7} \end{bmatrix}$$

the corresponding matrices obtained are:

$$\tilde{A} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 & 0 \\ 0 & 0 & 6 & 20 & 6 & 0 \\ 0 & 0 & 1 & 21 & 35 & 7 \end{bmatrix} \qquad \tilde{K} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{30} & \frac{1}{3} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -\frac{1}{12} & \frac{5}{12} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{42} & -\frac{1}{6} & \frac{1}{2} & \frac{1}{7} \end{bmatrix}$$

$$Q\vec{P} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 64 & 0 \\ 0 & 0 & 0 & 0 & 0 & 128 \end{bmatrix} \begin{bmatrix} p^2 \\ p^3 \\ p^4 \\ p^5 \\ p^6 \\ p^7 \end{bmatrix} = \begin{bmatrix} 4p^2 \\ 8p^3 \\ 16p^4 \\ 32p^5 \\ 64p^6 \\ 128p^7 \end{bmatrix} = \begin{bmatrix} u^2 \\ u^3 \\ u^4 \\ u^5 \\ u^6 \\ u^7 \end{bmatrix} = \vec{U}$$

2 Result from two studies by different authors

Nel 1986 A.W.F. Edwards found a result [3] that now we can exprime in that way:
$\vec{U} = 2\tilde{A}\vec{D}$ from witch

$$\vec{D} = \frac{1}{2}\tilde{A}^{-1}\vec{U}$$

Nel 2019 Nigel Derby found [4] that:

$$\vec{S} = 2\tilde{K}\vec{D}$$

replacing $\vec{D}$ is obtained:

$$\vec{S} = \tilde{K}\tilde{A}^{-1}\vec{U} = \tilde{K}\tilde{A}^{-1}Q\vec{P}$$

2.1 Exemples

$$\begin{bmatrix} S_1^2 \\ S_2^2 \\ S_3^2 \\ S_4^2 \\ S_5^2 \\ S_6^2 \end{bmatrix} = \tilde{K} \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 & 0 \\ 0 & 1 & 10 & 5 & 0 & 0 \\ 0 & 0 & 6 & 20 & 6 & 0 \\ 0 & 0 & 1 & 21 & 35 & 7 \end{bmatrix}^{-1} \begin{bmatrix} u^2 \\ u^3 \\ u^4 \\ u^5 \\ u^6 \\ u^7 \end{bmatrix}$$

$$\begin{bmatrix} S_1^2 \\ S_2^2 \\ S_3^2 \\ S_4^2 \\ S_5^2 \\ S_6^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & -\frac{1}{30} & \frac{1}{3} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -\frac{1}{12} & \frac{5}{12} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{42} & -\frac{1}{6} & \frac{1}{2} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{3}{10} & \frac{3}{5} & -\frac{1}{2} & \frac{1}{5} & 0 & 0 \\ \frac{5}{6} & -\frac{5}{3} & \frac{17}{12} & -\frac{2}{3} & \frac{1}{6} & 0 \\ -\frac{691}{210} & \frac{691}{15} & \frac{118}{21} & -\frac{41}{15} & -\frac{5}{6} & \frac{1}{7} \end{bmatrix} Q \begin{bmatrix} p^2 \\ p^3 \\ p^4 \\ p^5 \\ p^6 \\ p^7 \end{bmatrix}$$

$$\begin{bmatrix} S_1^2 \\ S_2^2 \\ S_3^2 \\ S_4^2 \\ S_5^2 \\ S_6^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{36} & \frac{1}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 & 0 & 0 \\ \frac{1}{900} & -\frac{1}{450} & -\frac{1}{60} & \frac{1}{25} & 0 & 0 \\ 0 & 0 & \frac{1}{144} & -\frac{1}{36} & \frac{1}{36} & 0 \\ \frac{1}{1764} & -\frac{1}{882} & -\frac{1}{196} & \frac{1}{42} & -\frac{1}{28} & \frac{1}{49} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 32 & 0 & 0 \\ 0 & 0 & 0 & 0 & 64 & 0 \\ 0 & 0 & 0 & 0 & 0 & 128 \end{bmatrix} \begin{bmatrix} p^2 \\ p^3 \\ p^4 \\ p^5 \\ p^6 \\ p^7 \end{bmatrix}$$

$$\begin{bmatrix} S_1^2 \\ S_2^2 \\ S_3^2 \\ S_4^2 \\ S_5^2 \\ S_6^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{9} & \frac{8}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{225} & -\frac{4}{225} & -\frac{4}{15} & \frac{32}{25} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & -\frac{8}{9} & \frac{16}{9} & 0 \\ \frac{1}{441} & -\frac{4}{441} & -\frac{4}{49} & \frac{16}{21} & -\frac{16}{7} & \frac{128}{49} \end{bmatrix} \begin{bmatrix} p^2 \\ p^3 \\ p^4 \\ p^5 \\ p^6 \\ p^7 \end{bmatrix}$$

from witch
$S_1^2 = p^2$
$S_2^2 = \frac{p^2+8p^3}{9}$
$S_3^2 = p^4$
$S_4^2 = \frac{p^2-4p^3-60p^4+288p^5}{225}$
$S_5^2 = \frac{p^4-8p^5+16p^6}{9} = \frac{(p^2-4p^3)^2}{9}$
$S_6^2 = \frac{p^2-4p^3-36p^4+336p^5-1008p^6+1152p^7}{441}$

3 The completed sequence

By applying the formula shown potentially you can complete the historical result obtained in 1631 by Faulhaber [1] who could express as a function of p only the sums S_i with odd index[2]
$S_1 = p$

$S_2 = \frac{\sqrt{p^2+8p^3}}{3}$
$S_3 = p^2$
$S_4 = \frac{\sqrt{p^2-4p^3-60p^4+288p^5}}{15}$
$S_5 = \frac{-p^2+4p^3}{3}$
$S_6 = \frac{\sqrt{p^2-4p^3-36p^4+336p^5-1008p^6+1152p^7}}{21}$
$S_7 = \frac{p^2-4p^3+6p^4}{3}$
$S_8 = \frac{\sqrt{9p^2-36p^3-300p^4+2832p^5-10400p^6+21120p^7-24000p^8+12800p^9}}{45}$
$S_9 = \frac{-3p^2+12p^3-20p^4+16p^5}{5}$
$S_{10} = \frac{\sqrt{25p^2-100p^3-820p^4+7760p^5-29136p^6+65472p^7-97152p^8+95744p^9-59136p^{10}+18432p^{11}}}{33}$
$\ldots$

3.1 Notes

- You can verify that $S_i(1) = 1$, $S_i(3) = 1 + 2^i$, ...
- The sequence of the denominators 1,3,1,15,3,21,3,45,5,33,... can be obtained from A064538 [5] by dividing each number in the sequence by two until the result become odd.
- If you use a spreadsheet, to translate a result expressed as in Figure 1, it is

	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	0,1111111111	0,8888888889	0	0	0	0	0	0	0	0
3	0	0	1	0	0	0	0	0	0	0
4	0,0044444444446	-0,01777777779	-0,2666666667	1,28	0	0	0	0	0	0
5	0	0	0,1111111111	-0,8888888889	1,777777778	0	0	0	0	0
6	0,002267573691	-0,00907029476	-0,0816326531	0,7619047619	-2,285714286	2,612244898	0	0	0	0
7	0	0	0,111111111	-0,8888888888	3,111111111	-5,333333333	4	0	0	0
8	0,0044444433326	-0,0177777733	-0,1481481558	1,398518526	-5,135802474	10,42962963	-11,85185185	6,320987654	0	0
9	-0,0000000743	0,000000029755	0,3599999491	-2,87999995	10,55999997	-23,03999999	31,36	-25,6	10,24	0
10	0,02295677097	-0,0918270839	-0,7529848693	7,125803961	-26,75482124	60,12121226	-89,21212126	87,91919193	-54,30303031	16,92561983

Figure 1: Polynomial coefficient matrix ($\tilde{K}\tilde{A}^{-1}Q$) obtained with a spreadsheet

convenient to multiply the obtained matrix on the left by a diagonal matrix having in the diagonal the square of the sequence of denominators. This way you obtain a matrix of integers that allows you to easily identify fractions corresponding to the various decimal expressions. For example, if the fourth row is multiplied by $15^2 = 225$, since 15 is the fourth value of the sequence, you obtain with good approximation $1, -4, -60, 288, 0, \ldots$ which allows us to deduce that the fourth row corresponds to $\frac{1}{225}, -\frac{4}{255}, -\frac{60}{255}, \frac{16}{255}, 0, \ldots$

References

[1] Frank J. Swetz and Victor J. Katz Johann, Mathematical treasures: Faulhaber's Accademiae Algebrae, MMA

[2] Donald Knuth, Johann Faulhaber and sums of powers, Internet Archive, 1992

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[5] The on-line encyclopedia of integer sequences, A064538, (1,2,6,4,30,12,42,24,90,20,66,24,2730,420...), N.J.A.Sloane, 1964

4 Fundig declaration

The author, born in Rome in 1950, declares to be a retired mathematics teacher, to be only an amateur researcher and to have no type of funding for this article. Sincerely Giorgio Pietrocola